Speculation and Leverage

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PRELIMINARY

March 19, 2014

Abstract

Speculative episodes typically involve leverage. For example, the well known tulipmania episode was accompanied by the introduction of foward contracts which allowed speculators to take leveraged positions in tulip bulbs. The Great Crash of 1929 was exacerbated by leveraged trusts which used leverage to buy stocks. These leveraged trusts could in turn be bought on margin which allowed speculators to hold highly leveraged positions. More recently, speculation in the housing market was accompanied by extreme leverage. In this paper, I provide a continuous time extension of the Harrison-Kreps(1978) speculative model with learning. Speculators have heterogeneous priors and learn about the unknown switching intensity between states. When the riskless rate is fixed, the speculative premium for each investor is determined by the expected present value of excess demand in the consumption and bond markets. Letting the riskless rate adjust so that the market for borrowing clears provides endogenous margin requirements which limit borrowing. In addition, the wealth distribution of speculators now becomes a determinant of speculative premia and a fairly rich set of speculative dynamics arise.

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1 Introduction

Recent events in financial markets suggest the importance of understanding how speculation and leverage are linked. However, the current literature on speculation typically considers partial equilibrium models in which the market for borrowing and lending does not clear – in other words there is more lending than borrowing or vice-versa. This approach is justified by either assuming infinitely wealthy speculators or unlimited borrowing at a fixed risk free rate. Either assumption implies speculators do not face capital limits nor do they face budget constraints. These types of assumptions may be reasonable when studying a small market but become less reasonable in trying to understand how recent events are linked to speculation and leverage.

Several papers have investigated speculation in financial markets. Miller(1978) presents a simple static model to argue that if there are impediments to short selling and investors have heterogeneous views on asset values, then only those who have the highest valuations will end up buying assets. As a result, assets will tend to be valued higher than the average valuation. While this is an important insight, the static nature of the argument and the partial equilibrium nature of the argument does not yield much insight into how, for example, interest rates might influence leverage through time.

Harrison and Kreps(1978) present a model of speculative investor behavior. In a model where again there are impediments to short selling, they show that even if an investor has a lower view of an asset's value, the right to resell the asset at a future date might provide a significant source of value to the investor and induce them to buy the asset with the intention of reselling it in the future. In this sense, they provide a formal model of the Keynsian notion of speculation. However, in their analysis they assume investors are infinitely wealthy which seems inconsistent with the basic economic principle of scarcity of resources.

In this paper I first sketch a simple extension of the Harrison and Kreps(1978) speculative model. While this extension is of independent interest, the primary motivation for this extension is to develop a suitable benchmark to examine how clearing the borrowing and lending market impacts speculative prices.

The basic uncertainty is generated by a simple (perhaps the simplest) continous time markov chain. For simplicity there are two states of nature possible at each time – low dividends or high dividends[1]. This leads to jumps in asset prices. While many papers have examined heterogeneous beliefs in a continuous information setting, this paper examines an equilibrium when disagreements are generated by jump uncertainty. Investors must post enough collateral to ensure that they do not jump to default. This stands in contrast to most continuous time equilibrium models with heterogeneous beliefs in the literature. Those models typically have continuous prices which effectively means that the equilibrium can be enforced with zero margin requirements.

At each point in time, risk neutral speculators disagree on the length of time the economy will stay in the current state. Speculators disagreements are generated by heterogenous priors. When investors do not learn through time as in the original Harrison Kreps model, the model has simple tractable solutions. As an extension we also consider the case where as time progresses, investors learn from the past and update their priors according to Bayes rule. The specification of the prior beliefs allows for investors to put more weight on the data or less weight on the data so the speed of learning can be investigated. In this model, speculators know which state they are in but are trying to estimate the unknown swithching intensity. This is different from models in which agents do not know the state they are in, but know the underlying parameters.

When investors have dogmatic beliefs and are infinitely wealthy or can borrow arbitrary amounts but face short sales constraints as in the original Harrison Kreps model, then fixing the mean belief, prices are increasing in the magnitude of the disagreement; this is expected based on the original Harrison Kreps analysis. However, the percentage price changes are *decreasing* as disagreement gets large. This is contrary to the common assertion that volatility should be raised by dispersion of beliefs. In particular, price volatility is highest when investors agree. To understand this effect, recall that in the Harrsion Kreps model prices reflect the private valuation plus the option to resell the asset. When the state shifts from high to low dividend, price changes reflect the impact of the change in private valuation to the investor selling the asset but the sale price is higher due to the speculative premium. This then lowers the percentage price change.

[1]Essentially, this can be thought of as a continuous time version of the model considered in Morris(1996). However, it is not a limiting case of that model.

While the model with Bayesian learners appears to not have tractable solutions, we can provide tight bounds on the prices using models of overconfidence which do have tractable solutions. In this case, the bounds show that prices converge to the investors' private valuations. The speed with which prices converge to private valuations depends on how much weight speculators put on the data and the magnitude of initial disagreement. These bounds can also be used to show the prices must converge to the correct prices in the limit.

More importantly, in the Harrison Kreps model, the speculative premium for each investor can be thought of as the expected discounted excess consumption for both investors over the dividend. The consumption market does not clear and the bond market doesn't clear. Each investor expects the other investor's wealth to grow unboundedly negative and this reflects borrowing to finance expected trading losses. In this sense the Harrison Kreps prices require that the market for borrowing and lending doesn't clear. Although the analysis in Loewenstein and Willard(2006) is concerned with violations of the law of one price which may or may not occur in this setting, we obtain a similar conclusion: the speculative premium in the Harrison Kreps model depends on markets not clearing. Violations of the law of one price can occur here as well, again due to the fact markets do not clear. This suggests the importance of examining these types of economies from a different perspective.

When speculators have limited capital many interesting results arise. For most parameters of our model, speculators can enjoy infinite expected utility when the other speculator sets the prices. As a result, it is never optimal for speculators to go bankrupt in equilibrium when they have limited capital. This is suprising because this occurs even when both investors learn from the data and will "eventually" agree on the true switching intensity between regimes.

The impact of limited capital provides several other interesting effects not found in speculative models where speculators have unlimited capital. Prices show pronounced trends punctuated by unpedictable jumps. Prices tend to be highest when speculators are all well capitalized. In this case investors have plenty of collateral to put on large speculative positions. However, as uncertainty unfolds, events tend to favor certain investors who accumulate more wealth. In contrast to the Harrison Kreps model, the wealthy investors become the marginal investor who sets prices. However, the less wealthy investors do not go bankrupt in our model. This is because prices set by the wealthy investors reflect very profitable opportunities for speculation. The less wealthy investors anticipate these prices and optimally reserve some wealth for this possibility. There are clear cycles of leveraging and de-leveraging in our model.

However, leverage does not perfectly track price levels. In particular, in the good state leverage is highest well after the peak prices occur. This is because volatility falls as prices fall beyond the peak. Because volatility gets lower, speculators can take highly levered positions without defaulting. When disagreements are very large, volatility can hit zero and we see phenomena similar to the Hart(1975) original example showing how then asset span can collapse.

There are dramatic effects in the term structure of interest rates. Just after the peak stock price is attained, the term structure of interest rates is steeply upward sloping. The term structure can also display clietele effects; the term structure need not be monotonic. This effect will be present in a model where investors disagree over rare events. However, learning will tend to smooth out this effect.

A recent literature examines speculation and default and links this activity to asset prices. Our model can help clarify and refine this analysis. First, speculative prices do not have any direct link to default. Recall speculators do not default in our equilibrium. In fact it is the lack of default which supports speculative prices. Second, our model indicates there could be long periods in which there is seemingly no speculative activity. During this time one group builds capital which allows progressively larger positions. At a certain point prices rise dramatically. In effect, when speculators do not eventually agree the economy will cycle through periods of little speculative activity followed by a dramatic price rise.

Even though agents never face a binding wealth constraint, in the very long run only one group of agents will survive. When agents have dogmatic beliefs, the agent with the correct beliefs survives. In contrast to Blume and Easely(2006), where all Bayesian learners survive in the limit, I show that in the case of risk neutral speculators, survival depends on the ratio of the prior densities evaluated at the truth and the initial wealth distribution. This difference is due to the fact that our agents have finite marginal utility at zero consumption, while in Blume and Easely agents have infinite marginal utility at zero consumption.

The paper is organized as follows. The next section describes the basic model and analyzes the partial equilibrium Harison Kreps model of speculation. Section 3 analyzes the model when markets clear while Section 4 contains results on long run survival of agents. Section 5 concludes.

2 A Simple Continous Time Limit of the Harrison Kreps Model

Here is a model which is a continuous time limit of the discrete time analysis in Harrison and Kreps(1978). The continuous time limit also offers new insight into the speculative model of Harrison and Kreps(1978).

Our model begins with a probability space $(\Omega, \mathcal{F}, P)$ on which a right continuous counting process N_t is defined. Information is revealed according to the right continuous filtration $\{\mathcal{F}_t\}_{t=0}^{\infty}$ generated by the counting process.

I consider a market for a a dividend paying asset (one share of stock) and riskless borrowing and lending. Trade occurs continuously over an infinite horizon. The dividend paying asset pays an instantaneous dividend at each point in time of $\delta_t dt$ so the cumulative dividends paid over an interval $[0, t]$ are given by $\int_0^t \delta_s ds$. Investors can also borrow and lend but face short sales constraints on the dividend paying asset. The evolution of the dividend is described by a simple regime shifting model. For simplicity assume there are two possible states of the world (0 and 1) at each time. In state 1, $\delta_t(1) = 1$ and in state 0, $\delta_t(0) = 0$. The regime shifts whenever the counting process N_t jumps.

In the "true" model the switch from State 0 to State 1 and back is governed by a Poisson process N_t with constant intensity λ. It is standard to assume that $N_0 = 0$. At each jump the state changes. Well known results on the Poisson process (see Karlin and Taylor(1975)) give

$$P[N_t = n] = e^{-\lambda t}\frac{(\lambda t)^n}{n!} \tag{2.1}$$

$$P[N_t \ \text{ odd}] = \frac{1}{2} - \frac{1}{2}e^{-2\lambda t} \tag{2.2}$$

$$P[N_t \ \text{ even}] = \frac{1}{2} + \frac{1}{2}e^{-2\lambda t} \tag{2.3}$$

There are two investor classes who differ in their prior probability distributions on λ which we denote by investor class A and investor class B. Each investor class $i = A, B$ would like to maximize the expected value of discounted payoffs from trade

$$E^i\left[\int_0^\infty e^{-\gamma t} c_t dt\right]$$

where c_t represents the nonnegative payoffs from a particular trading strategy.

Investors have prior beliefs on λ which are Gamma distributed, that is the prior probability density for λ for investor class i is given by

$$\frac{k_i^{z_i} e^{-k_i\lambda}\lambda^{z_i-1}}{\Gamma(z_i)} \quad i = A,\ B \tag{2.4}$$

where $\Gamma(z) = \int_0^\infty e^{-t}t^{z-1}dt$. If $k_A \neq k_B$ and/or $z_A \neq z_B$ then agents have heterogeneous priors. Each investor class updates their beliefs given the data using Bayes rule. Thus at time t given $N_t = n$ their posterior belief on λ is given by

$$E^i[\lambda|N_t = n] = \frac{\int_0^\infty \lambda^{z_i} e^{-k_i\lambda} e^{-\lambda t}\frac{(\lambda t)^n}{n!}d\lambda}{\int_0^\infty \lambda^{z_i-1} e^{-k_i\lambda} e^{-\lambda t}\frac{(\lambda t)^n}{n!}d\lambda} = \frac{z_i + n}{t + k_i} \equiv \lambda_i(n,t) \tag{2.5}$$

In particular

$$E^i[\lambda] = \frac{z_i}{k_i} \tag{2.6}$$

and the process $\lambda_i(N_t, t)$ is a P^i martingale. The process $N_t - \int_0^t \lambda_i(N_s, s)ds$ is a P^i martingale.[2] $\lambda_i(n,t)$ is decreasing in t and increasing in N. After an infinite amount of data, $\lambda_i(N_t, t) \to \frac{N_t}{t} \to \lambda$ so in the limit agents learn the truth.

Similar calculations give for $s \leq t$ and $n \geq m$

$$\begin{aligned} P^i[N_t = n|N_s = m] &= \frac{\int_0^\infty \lambda^{z_i-1} e^{-k_i\lambda} e^{-\lambda(t-s)}\frac{(\lambda(t-s))^{n-m}}{(n-m)!} e^{-\lambda s}\frac{(\lambda s)^m}{m!}d\lambda}{\int_0^\infty \lambda^{z_i-1} e^{-k_i\lambda} e^{-\lambda s}\frac{(\lambda s)^m}{m!}d\lambda} \\ &= \frac{(t-s)^{n-m}}{(n-m)!}\frac{\Gamma(n+z_i)}{\Gamma(m+z_i)}\frac{(s+k_i)^{z_i+m}}{(t+k_i)^{n+z_i}} \end{aligned} \tag{2.7}$$

If n is even we have

$$P^i[N_t \ \text{odd}|N_s = n] = \frac{1}{2} - \frac{1}{2}\frac{(s+k_i)^{m+z_i}}{(2(t-s)+s+k_i)^{z_i+m}} \tag{2.8}$$

$$P^i[N_t \ \text{even}|N_s = n] = \frac{1}{2} + \frac{1}{2}\frac{(s+k_i)^{z_i+m}}{(2(t-s)+s+k_i)^{z_i+m}} \tag{2.9}$$

[2] To be precise we should write $N_t - \int_0^t \lambda_i(N_{s-}, s)ds$. However, since $\lambda(N_t, t) = \lambda(N_{t-}, t)$ ℓ almost surely these are the same processes.

If n is odd

$$P^i[N_t \ \text{ odd}|N_s = n] = \frac{1}{2} + \frac{1}{2}\frac{(s+k_i)^{m+z_i}}{(2(t-s)+s+k_i)^{z_i+m}} \tag{2.10}$$

$$P^i[N_t \ \text{ even}|N_s = n] = \frac{1}{2} - \frac{1}{2}\frac{(s+k_i)^{z_i+m}}{(2(t-s)+s+k_i)^{z_i+m}} \tag{2.11}$$

and

$$P^i[N_t \ \text{ odd}] = \frac{1}{2} - \frac{1}{2}\frac{k_i^{z_i}}{(2t+k_i)^{z_i}} \tag{2.12}$$

$$P^i[N_t \ \text{ even}] = \frac{1}{2} + \frac{1}{2}\frac{k_i^{z_i}}{(2t+k_i)^{z_i}} \tag{2.13}$$

and in particular

$$P^i[N_t = n] = \frac{t^n}{n!}\frac{\Gamma(n+z_i)}{\Gamma(z_i)}\frac{k_i^{z_i}}{(t+k_i)^{n+z_i}} \tag{2.14}$$

The parameters z_i and k_i influence the prior belief on λ and how much weight the investor puts on the data when updating the beliefs.[3] Recall initial expected value of λ is given by $\lambda_i(0,0) = \frac{z_i}{k_i}$. Fixing the prior belief λ_i and setting $z_i \equiv k_i\lambda_i$ and letting $k_i \to \infty$ we obtain the limiting case where the investor puts no weight on the data and does not learn. The probability distribution is given by

$$P^i[N_t = n] = e^{-\lambda_i t}\frac{(\lambda_i t)^n}{n!} \tag{2.15}$$

This specification assumes investors do not update their beliefs no matter how much data they observe, in other words they have dogmatic beliefs.

On the other hand, letting z_i and k_i go to zero, the investor puts more weight on the data and his posterior estimate of λ comes close to the empirical frequency $\frac{N_t}{t}$. Thus our choice of prior beliefs represents a fairly general class of priors which can accomodate different beliefs as well as differential learning.

We can compute the private valuation of the stock dividends for an investor under the assumption there is no trade. We denote this private valuation by $S^A(0,n,t)$ for investor class A in state 0, $S^A(1,n,t)$ for investor class A in state 1,$S^B(0,n,t)$ for investor class B in state 0, and $S^B(1,n,t)$ for investor class B in state 1. Simple computations give the expected present value of dividends for each investor class in

[3] z_i is the shape parameter and $\frac{1}{k_i}$ is the scale parameter. The variance of λ is equal to $\frac{z_i}{k_i^2}$.

each state:

$$\begin{aligned}
S^i(0,n,t) = E^i\left[\int_t^\infty e^{-\gamma(s-t)}\delta_s ds | N_t = n\right] &= \int_t^\infty e^{-\gamma(s-t)} P^i[N_s - n \ \ \text{odd} | N_t = n] ds \\
&= \int_t^\infty e^{-\gamma(s-t)} \left(\frac{1}{2} - \frac{1}{2}\frac{(t+k_i)^{z_i+n}}{(2(s-t)+t+k_i)^{z_i+n}}\right) ds \\
&= \frac{1}{2\gamma}\left(1 - (t+k_i)^{n+z_i} e^{\frac{\gamma(t+k_i)}{2}} \left(\frac{\gamma}{2}\right)^{n+z_i} \Gamma(1 - z_i - n, \frac{\gamma(t+k_i)}{2})\right) \quad (2.16)
\end{aligned}$$

$$\begin{aligned}
S^i(1,n,t) = E^i\left[\int_t^\infty e^{-\gamma(s-t)}\delta_s ds | N_t = n\right] &= \int_t^\infty e^{-\gamma(s-t)} P^i[N_s - n \ \ \text{even} | N_t = n] ds \\
&= \int_t^\infty e^{-\gamma(s-t)} \left(\frac{1}{2} + \frac{1}{2}\frac{(t+k_i)^{z_i+n}}{(2(s-t)+t+k_i)^{z_i+n}}\right) ds \\
&= \frac{1}{2\gamma}\left(1 + (t+k_i)^{n+z_i} e^{\frac{\gamma(t+k_i)}{2}} \left(\frac{\gamma}{2}\right)^{n+z_i} \Gamma(1 - z_i - n, \frac{\gamma(t+k_i)}{2})\right) \quad (2.17)
\end{aligned}$$

where $\Gamma(z,x) = \int_x^\infty e^{-t}t^{z-1}dt$. We note we have the following limits

$$\lim_{t\to\infty} S^i(1,n,t) = \frac{1}{\gamma} \quad \lim_{t\to\infty} S^i(0,n,t) = 0 \tag{2.18}$$

$$\lim_{n\to\infty} S^i(\cdot,n,t) = \frac{1}{2\gamma} \tag{2.19}$$

When the investors have dogmatic beliefs we can compute the private value of the dividends as

$$S^i(1,n,t) = \int_0^\infty e^{-\gamma t} P^i\{N_t - n \ \ \text{even}\} dt = \frac{\lambda_i + \gamma}{\gamma(\gamma + 2\lambda_i)} \tag{2.20}$$

$$S^i(0,n,t) = \int_0^\infty e^{-\gamma t} P^i\{N_t - n \ \ \text{odd}\} dt = \frac{\lambda_i}{\gamma(\gamma + 2\lambda_i)} \tag{2.21}$$

The following proposition documents how the private valuations compare when there is learning versus when there is no-learning.

Proposition 2.1. *Each individual's private valuation satisfies*

$$\frac{\gamma + \lambda_i(n+2,t)}{\gamma(\gamma + 2\lambda_i(n+1,t))} > S^i(1,n,t) > \frac{\lambda_i(n,t) + \gamma}{\gamma(\gamma + 2\lambda_i(n,t))} \tag{2.22}$$

$$\frac{\lambda_i(n,t)}{\gamma(\gamma + 2\lambda_i(n+1,t))} < S^i(0,n,t) < \frac{\lambda_i(n,t)}{\gamma(\gamma + 2\lambda_i(n,t))} \tag{2.23}$$

Proposition 2.1 says in state 1 the private valuation is higher than in the case with no learning and in state 0 the private valuation is lower than the case with no learning. This is because while each investor class has a prior belief on the length of time spent in the current regime, conditional on surviving for that long the investor will revise their expectations upward. Therefore learning makes the stock more valuable in State 1 and less valuable in State 0 than in the no learning case.

2.1 Trade

When we allow trade in the risky asset, a speculative premium arises in the sense that agents will pay more than their private valuation because they anticipate selling the asset in the future for an inflated value. Here we show the equilibrium in the Harrison Kreps setting, assuming infinitely wealthly investors, or unlimited borrowing and lending at the continuously compounded rate γ. Define $R_t = e^{\gamma t}$.

Choice Problem 2.1 (Choice Problem In Harrison Kreps)**.** *Given securities endowments θ^i, choose predictable*[4] *securities holdings θ_t and α_t and adapted consumption c_t to maximize*

$$E^i\left[\int_0^\infty e^{-\gamma t} c_t dt\right]$$

subject to

$$dW_t = \alpha_t dR_t + \theta_t dS_t + \theta_t \delta_t dt - c_t dt$$

$$c_t \geq 0, \quad \theta_t \geq \underline{\theta},$$

and

$$\lim_{t\to\infty} E^i\left[e^{-\gamma t} W_t\right] = 0$$

where $W_t \equiv \alpha_t R_t + \theta_t S_t$ and $W_0 = \theta^i S_0$.

The next proposition provides a necessary condition for an optimal solution. It says for an investor to have an optimal solution, the investor's expected stock return cannot exceed γ.

Proposition 2.2. *A necessary condition for an optimal solution is the process $e^{-\gamma t} S_t + \int_0^t e^{-\gamma s}\delta_s ds$ is a P^i supermartingale. Therefore in any Harrison Kreps equilibrium, there is a speculative premium: $S(\iota, N, t) \geq S^i(\iota, N, t)$ for each i.*

[4]That is, adapted to the sigma algebra generated by all adapted left continuous processes. This ensures the portfolio choice is performed prior to the realization of the jumps.

2.1.1 Harrison Kreps Equilibrium

We now define the equilibrium in the Harrison Kreps model. In this equilibrium the risk free rate is fixed, agents optimize, and the market for the stock clears.

Definition 2.1. *A Harrison Kreps equilibrium is a stock price $S(\iota, N, t)$, $\iota = 0, 1$, such that given this price agents solve the choice problem 2.1 and the market for stock clears: $\theta_t^A + \theta_t^B = 1$ $\ell \otimes P^i$ almost surely where θ_t^i is the solution to choice problem 2.1.*

We begin with a fairly general specification of stock price

$$dS_t = (\gamma S_{t-} - \delta_t - \lambda_t^Q \Delta S_t)dt + \Delta S_t dN_t \tag{2.24}$$

where $\Delta S_t \equiv S(1 - \iota_t, N_t + 1, t) - S(\iota_t, N, t)$. The discounted stock price plus discounted cumulative dividend

$$de^{-\gamma t} S_t + e^{-\gamma t} \delta_t dt = e^{-\gamma t} \Delta S_t \left(dN_t - \lambda_t^Q dt \right) \tag{2.25}$$

is a Q martingale under a probability measure for which the counting process N_t has intensity λ_t^Q. This can be rewritten

$$de^{-\gamma t} S_t + e^{-\gamma t} \delta_t dt = e^{-\gamma t} (\lambda_i(N_t, t) - \lambda_t^Q) \Delta S_t dt + e^{-\gamma t} \Delta S_t \left(dN_t - \lambda_i(N_t, t) dt \right) \tag{2.26}$$

Recall $N_t - \int_0^t \lambda_i(N_s, s) ds$ is a P^i martingale. The requirement that this is a P^i supermartingale for each $i = A, B$ amounts to

$$(\lambda_i(N_t, t) - \lambda_t^Q) \Delta S_t \leq 0 \quad i = A, B \tag{2.27}$$

Market clearing in the stock means

$$(\lambda_i(N_t, t) - \lambda_t^Q) \Delta S_t = 0 \tag{2.28}$$

for some i. Therefore to clear the stock market we must have

$$\lambda_t^Q \equiv \lambda^Q(N_t, t) = \max_{\{i\}} [\lambda_i(N_t, t)] \quad \text{when} \quad \Delta S_t > 0 \tag{2.29}$$

$$\lambda_t^Q \equiv \lambda^Q(N_t, t) = \min_{\{i\}} [\lambda_i(N_t, t)] \quad \text{when} \quad \Delta S_t < 0 \tag{2.30}$$

This intuitively says in state 0, the investor who has the quickest estimate of when the stock will revert to paying a dividend will value the stock highest and in state 1, the investor who has the slowest estimate of when the state will change will value the stock the highest.

It follows that

$$\frac{\partial S_t}{\partial t} = (\gamma S_{t-} - \delta_t - \lambda_t^Q \Delta S_t) = ((\gamma + \lambda_t^Q) S(\iota_t, N_t, t) - \delta_t - \lambda_t^Q S(1 - \iota_t, N_t + 1, t) \quad (2.31)$$

and

$$S(1, t, n) = \int_0^\infty e^{-\gamma s} \lambda^Q(n, s+t) \exp\left(-\int_0^s \lambda^Q(n, u+t) du\right) \left(\frac{1}{\lambda^Q(n, s+t)} + S(0, s+t, n+1)\right) ds + \lim_{u \to \infty} e^{-\gamma u} S(0, u+t, n+1) \quad (2.32)$$

$$S(0, t, n) = \int_0^\infty e^{-\gamma s} \lambda^Q(n, s+t) \exp\left(-\int_0^s \lambda^Q(n, u+t) du\right) S(1, s+t, n+1) ds + \lim_{u \to \infty} e^{-\gamma u} S(1, u+t, n+1) \quad (2.33)$$

In general there can be many solutions to equations (2.32) and (2.33). For example, for any function M such that $\frac{\partial M(n,t)}{\partial t} = -\lambda^Q(n,t)(M(n+1,t) - M(n,t))$, then one can add $e^{\gamma t} M(n,t)$ to any solution to produce a new solution. These solutions can thus have asset pricing bubbles in the sense that the stock price is higher than a portfolio trading strategy which replicates its dividends.

Proposition 2.3. *Assume $\lambda_A(N,t) > \lambda_B(N,t)$ for all N and t, and define $\lambda^Q(N,t) = \lambda_A(N,t)$ in state 0 and $\lambda^Q(N,t) = \lambda_B(N,t)$ in state 1. Given any nonnegative solution to equations (2.32) and (2.33) with $\Delta S(1,N,t) < 0$ and $\Delta S(0,N,t) > 0$, these are potential prices in a Harrison Kreps equilibrium. When $\underline{\theta} = 0$, the associated equilibrium trading strategies are for investor class A to buy the stock in state 0 and hold the riskless asset in state 1 and for investor class B to buy the stock in state 1 and hold the riskless asset in state 0. Any consumption policy for which*

$$E^i \left[\int_0^\infty e^{-\gamma s} c_s^i ds\right] = \theta_i S(\iota, 0, 0) \quad (2.34)$$

is optimal for these strategies. Moreover, the equilibrium stock price satisfies $S(\iota, N, t) > S^i(\iota, N, t)$, that is the speculative premium is strictly positive for each i.

2.1.2 Asset Pricing Bubbles

One consequence of not clearing markets is that there are many potential equilibrium prices due to asset pricing bubbles. However, in absence of these types of bubbles we should have $0 \leq S(\cdot, n, t) \leq \frac{1}{\gamma}$. Therefore it seems natural to focus on bounded solutions. Harrison Kreps call these prices *minimal consistent* prices. The importance of these prices is that the stock price is equal to the expected discounted value of future dividends under the Q measure. In other words

$$S^Q(\iota, N, t) = E^Q\left[\int_t^\infty e^{-\gamma(s-t)}\delta_s ds|\mathcal{F}_t\right] \tag{2.35}$$

where E^Q is the expectation operator for a probability measure under which the intensity of the counting process is $\lambda^Q(N,t)$. In general for any Harrison Kreps equilibrium

$$S(\iota, N, t) = E^Q\left[\int_t^\infty e^{-\gamma(s-t)}\delta_s ds|\mathcal{F}_t\right] + \lim_{t\to\infty} E^Q\left[e^{-\gamma t}S_t|\mathcal{F}_t\right] \tag{2.36}$$

and defining the bubble component, $B_t = S_t - S_t^Q$ we have

$$\begin{aligned} dB(\iota, N_t, t) = \{&\gamma B(\iota, N_t, t) - \lambda^Q(N_t, t)(B(1-\iota, N_t+1, t) - B(\iota, N_t, t))\}dt \\ &+ (B(1-\iota, N_t+1, t) - B(\iota, N_t, t))dN_t \end{aligned} \tag{2.37}$$

Comparing to the dynamics of a wealth process which invests positive amounts in the stock and never consumes, we see that one can think of the bubble component as a strategy which tranfers consumption to "infinity." We can thus take any of these wealth processes and add it to S^Q to produce another equilibrium price in the Harrison Kreps model. The next proposition illustrates the consequences of this.

Proposition 2.4. *Suppose in the equilibrium described in Proposition 2.3,*

$$\lim_{t\to\infty} E^Q\left[e^{-\gamma t}S_t\right] > 0. \tag{2.38}$$

Then there is a violation of the law of one price, that is the stock price is higher than the initial value of a portfolio trading strategy which replicates the stock's dividends and maintains non-negative wealth. If $\Delta S^Q(1, N, t) < 0$ and $\Delta S^Q(0, N, t) > 0$, portfolio strategy does not involve short sales.

The violation of the law of one price requires infinite retrade to support it. An equilibrium price can violate the law of one price even when there is no disagreement. This can happen because we are not clearing the market for the riskless asset. Later, Proposition 2.10 shows the link between violations of the law of one price and lack of market clearing.

2.1.3 Minimal Consistent Prices When Investors have Dogmatic Beliefs

In general Equations (2.32) and (2.33) must be solved recursively. In the special case of no learning we have explicit equations. These expressions are also a useful benchmark to interpret the case with learning studied in the next section.

Proposition 2.5. *Suppose investors have dogmatic beliefs and* $\lambda_A > \lambda_B$. *The minimal consistent equilibrium stock price in the Harrison Kreps model is given by*

$$S^Q(0, N, t) = \frac{\lambda_A}{\gamma(\gamma + \lambda_A + \lambda_B)} \tag{2.39}$$

$$S^Q(1, N, t) = \frac{\lambda_A + \gamma}{\gamma(\gamma + \lambda_A + \lambda_B)} \tag{2.40}$$

and the equilibrium stock price is independent of N *and* t.

Stock price percentage jumps are given by

$$\frac{\Delta S^Q(0, N, t)}{S^Q(0, N, t)} = \frac{\gamma}{\lambda_A} \tag{2.41}$$

$$\frac{\Delta S^Q(1, N, t)}{S^Q(1, N, t)} = -\frac{\gamma}{\lambda_A + \gamma} \tag{2.42}$$

Stock price volatility is related to the jump magnitudes and of course the "true" value of the counting process intensity. The jump magnitudes are determined by the investor with the highest λ_i. As a result, fixing the mean belief, prices go up and volatility decreases as beliefs diverge. Notice that this also implies the volatility of prices is lower when the speculative premium is highest. The intuition for this result is that as beliefs diverge, a greater portion of the value is due to the resale of the stock and not fundamentals. This then buffers the stock price from changes in the fundamentals. Also it is worth noting that these results do not depend on how many speculators there are and their beliefs; everything is determined by the highest and lowest λ.

As observed in Harrison and Kreps(1978) the equilibrium gives rise to a speculative premium for each investor, the equilibrium stock price is higher than each investor's private valuation of the dividends: when investors do not learn from the data we have

$$S(1, N, t) - S^i(1, N, t) \geq \frac{\lambda_A + \gamma}{\gamma(\gamma + \lambda_A + \lambda_B)} - \frac{\lambda_i + \gamma}{\gamma(\gamma + 2\lambda_i)} \tag{2.43}$$

$$S(0, N, t) - S^i(0, N, t) \geq \frac{\lambda_A}{\gamma(\gamma + \lambda_A + \lambda_B)} - \frac{\lambda_i}{\gamma(\gamma + 2\lambda_i)} \tag{2.44}$$

Case	S(1,0,0)	S(0,0,0)	Δ S/S(1)	Δ S/S(0)
$S^A : \lambda_A = 0.6$	5.38	4.62	-0.14	0.17
$S^B : \lambda_B = 0.5$	5.45	4.55	-0.17	0.20
$S^Q : \lambda_A = 0.6, \lambda_B = 0.5$	5.83	5	-0.14	0.17
$S^A : \lambda_A = 1$	5.23	4.76	-0.09	0.10
$S^Q : \lambda_A = 1, \lambda_B = 0.5$	6.85	6.25	-0.09	0.10

Table 1: Comparison of Valuation.
This table shows the private vaulations for Investor Class A and B and the Harrison Kreps equilibrium prices. All cases use $\gamma = 0.1$.

with equality when the equilibrium stock price is the minimal consistent price. Table 1 displays the values for the minimal consistent equilibrium prices in various settings.

Remark 2.1. *Our results here are easily generalized to the case where $\delta(1) = H$ and $\delta(0) = L$. Then the minimal consistent prices are given by*

$$(H - L)S^Q(\iota, N_t, t) + \frac{L}{\gamma} \tag{2.45}$$

where $S^Q(\iota, N, t)$ the minimal consistent price above.

2.1.4 A Comparative Static Result and Bounds on Minimal Consistent Prices

In this section we establish a simple comparative static result and use this to bound the prices in the Harrison Kreps equilibrium.

Proposition 2.6. *Assume $\lambda_A(N,t) \geq \lambda_B(N,t)$ for all $t \geq t^*$ and $N \geq N^*$. Let $\tilde{S}^Q$ be the minimal consistent prices when investor class A estimates the intensity to be $\tilde{\lambda}_A(N,t)$ and investor class B estimates the intensity to be $\tilde{\lambda}_B(N,t)$. Then if $\lambda_A(N,t) \leq \tilde{\lambda}_A(N,t)$ and $\lambda_B(N,t) \geq \tilde{\lambda}_B(N,t)$,*

$$\tilde{S}^Q(\iota, N, t) \geq S^Q(\iota, N, t) \quad \text{for } t \geq t^* \text{and } N \geq N^* \tag{2.46}$$

Proposition 2.6 says that increasing disagreement causes the Harrison Kreps minimal consistent prices to be higher in both states; in terms of the underlying parameters we have the following corollary.

Corollary 2.1. *Assume $k_A < k_B$ and $z_A > z_B$. The minimal consistent prices are increasing in z_A and k_B and decreasing in z_B and k_A.*

We now use Proposition 2.6 to bound the Harrison Kreps equilibrium minimal consistent prices. To do this we modify our model so that $\lambda_B(N,t)$ is as before but now investor class A believes the true intensity to be

$$\tilde{\lambda}_A(N,t) = A + B\lambda_B(N,t) \tag{2.47}$$

This implies Investor class A believes the true intensity is $A+B\lambda$ where λ is distributed according to a Gamma distribution only now with shape and scale parameters z_B and $\frac{1}{k_B}$. In other words, both investor classes agree on the distribution of λ but they disagree on the values A and B. In this setting the private valuation for investor class B is as before while the private valuation for investor class A in state 1 is given by

$$\begin{aligned}
\tilde{S}^A(1,N,t) &= \frac{\int_0^\infty \frac{\gamma+A+B\lambda}{\gamma(\gamma+2(B\lambda+A))} k_B^{z_B} e^{-k_B\lambda}\lambda^{z_B-1}e^{-\lambda t}\frac{(\lambda t)^N}{N!}d\lambda}{\int_0^\infty k_B^{z_B} e^{-k_B\lambda}\lambda^{z_B-1}e^{-\lambda t}\frac{(\lambda t)^N}{N!}d\lambda} \\
&= \frac{1}{\gamma}e^{\frac{(2A+\gamma)(k_B+t)}{2B}}\left(\frac{(2A+\gamma)(k_B+t)}{2B}\right)^{z_B+N} \\
&\times\left(\frac{A+\gamma}{2A+\gamma}\Gamma\left(1-z_B-N,\frac{(2A+\gamma)(k_B+t)}{2B}\right)+\frac{z_B+N}{2}\Gamma\left(-z_B-N,\frac{(2A+\gamma)(k_B+t)}{2B}\right)\right) \\
&= \frac{1}{\gamma}e^{\frac{(2A+\gamma)(k_B+t)}{2B}}\left(\frac{(2A+\gamma)(k_B+t)}{2B}\right)^{z_B+N}\left(\frac{A+\gamma}{2A+\gamma}-\frac{1}{2}\right)\Gamma\left(1-z_B-N,\frac{(2A+\gamma)(k_B+t)}{2B}\right)+\frac{1}{2\gamma}
\end{aligned} \tag{2.48}$$

where we use the recursion (see the NIST Digital Library of Mathematical Functions)

$$\Gamma(a+1,x) = a\Gamma(a,x) + x^a e^{-x} \tag{2.49}$$

A similar exercise gives the private valuation for state 0.

$$\begin{aligned}
\tilde{S}^A(0,N,t) &= \frac{\int_0^\infty \frac{A+B\lambda}{\gamma(\gamma+2(B\lambda+A))} k_B^{z_B} e^{-k_B\lambda}\lambda^{z_B-1}e^{-\lambda t}\frac{(\lambda t)^N}{N!}d\lambda}{\int_0^\infty k_B^{z_B} e^{-k_B\lambda}\lambda^{z_B-1}e^{-\lambda t}\frac{(\lambda t)^N}{N!}d\lambda} \\
&= \frac{1}{\gamma}e^{\frac{(2A+\gamma)(k_B+t)}{2B}}\left(\frac{(2A+\gamma)(k_B+t)}{2B}\right)^{z_B+N} \\
&\times\left(\frac{A}{2A+\gamma}\Gamma\left(1-z_B-N,\frac{(2A+\gamma)(k_B+t)}{2B}\right)+\frac{z_B+N}{2}\Gamma\left(-z_B-N,\frac{(2A+\gamma)(k_B+t)}{2B}\right)\right) \\
&= \frac{1}{\gamma}e^{\frac{(2A+\gamma)(k_B+t)}{2B}}\left(\frac{(2A+\gamma)(k_B+t)}{2B}\right)^{z_B+N}\left(\frac{A}{2A+\gamma}-\frac{1}{2}\right)\Gamma\left(1-z_B-N,\frac{(2A+\gamma)(k_B+t)}{2B}\right)+\frac{1}{2\gamma}
\end{aligned} \tag{2.50}$$

Notice that when $A=0$ and $B=1$ this is just the private valuation for investor B and the private valuation satisfies the bounds (see Proposition 2.1)

$$\frac{\gamma+\tilde{\lambda}_A(N+1,t)+\frac{\gamma B}{(2A+\gamma)(t+k_B)}}{\gamma(\gamma+2\tilde{\lambda}_A(N+1,t))} > \tilde{S}^A(1,N,t) > \frac{\gamma+\tilde{\lambda}_A(N,t)}{\gamma(\gamma+2\tilde{\lambda}_A(N,t))} \tag{2.51}$$

$$\frac{\tilde{\lambda}_A(N+1,t) - \frac{\gamma B}{(k_B+t)(2A+\gamma)}}{\gamma(\gamma + 2\tilde{\lambda}_A(N+1,t))} < \tilde{S}^A(0,N,t) < \frac{\tilde{\lambda}_A(N,t)}{\gamma(\gamma+2\tilde{\lambda}_A(N,t))} \tag{2.52}$$

These inequalities imply that the private valuation in state 1 are always higher then the case where investor A does not change their estimate of the switching intensity. However, in state 0, the private valuation is always lower than the case where investor A does not change their estimate of the switching intensity.

While this model is of independent interest in that it can be thought of as a model of overconfidence as in Scheinkman and Xiong(2003), the advantage of this model is it gives explicit closed form solutions. Choosing $A = \frac{z_A - z_B}{k_A}$ and $B = \frac{k_B}{k_A}$ we have $\lambda_A(N,t) \leq \tilde{\lambda}_A(N,t)$. This is then useful because from Proposition 2.6 prices in this set-up provide upper bounds for the prices in the Harrison Kreps equilibrium with Bayesian learners with priors given by heterogeneous gamma distributions. The following proposition gives tighter bounds than this.

Proposition 2.7. *Assume $z_a \geq z_B$ and $k_A \leq k_B$. Let $A(t,N) = \frac{z_A - z_B}{t+k_A}$ and $B(t) = \frac{k_B+t}{k_A+t}$. Then, if $z_A \leq z_B + \gamma(k_B+t)$, the minimal consistent prices in the Harrison Kreps equilibrium admit the bounds $S^Q(\iota,N,t) \leq \tilde{S}^Q(\iota,N,t)$ where*

$$\tilde{S}^Q(1,N,t) = \frac{1}{\gamma(B(t)+1)} e^{X(t,N)} \left(X(t,N)\right)^{z_B+N} \Gamma\left(1 - z_B - N, X(t,N)\right) + \frac{B(t)}{\gamma(B(t)+1)} \tag{2.53}$$

$$\tilde{S}^Q(0,N,t) = \frac{1}{\gamma(1+B(t))} e^{X(t,N)} \left(X(t,N)\right)^{z_B+N} \left(\frac{A(t,N) - \gamma B(t)}{A(t,N)+\gamma}\right) \Gamma\left(1 - z_B - N, X(t,N)\right) + \frac{B(t)}{\gamma(B(t)+1)} \tag{2.54}$$

and

$$X(t,N) = \frac{(A(t,N)+\gamma)(k_B+t)}{B(t)+1}. \tag{2.55}$$

In addition, these bounds admit the bounds

$$\frac{\gamma + \tilde{\lambda}_A(N+2,t)}{\gamma(\gamma + \tilde{\lambda}_A(N+1,t) + \lambda_B(N+1,t))} > \tilde{S}^Q(1,N,t) > \frac{\gamma + \tilde{\lambda}_A(N,t)}{\gamma(\gamma + \tilde{\lambda}_A(N,t) + \lambda_B(N,t))} \tag{2.56}$$

and

$$\frac{\tilde{\lambda}_A(N+1,t) + \frac{A(t,N) - \gamma B(t)}{(k_B+t)(A(t,N)+\gamma)}}{\gamma(\gamma + \tilde{\lambda}_A(N+1,t) + \lambda_B(N+1,t))} < \tilde{S}^Q(0,N,t) < \frac{\tilde{\lambda}_A(N,t)}{\gamma(\gamma + \tilde{\lambda}_A(N,t) + \lambda_B(N,t))} \tag{2.57}$$

where $\tilde{\lambda}_A(N,t) = A(t,N) + B(t)\lambda_B(N,t)$

The prices $\tilde{S}$ in the proposition do not correspond to an equilibrium price system but correspond to the time t values of the Harrison Kreps minimal consistent prices for different economies where $\tilde{\lambda}_A(N+M, s+t) = A(t,N) + B(t)\lambda_B(N+M, s+t)$ for $s \geq 0$ and $M \geq 0$. The restriction $z_A \leq z_B + \gamma(k_B + t)$ is not, strictly speaking, necessary; if it doesn't hold, the inequalities in Equations (2.57) reverse and the prices in the different economies can have some interesting, but unusual, behavior[5]. However, in the limit the restriction will hold so we restrict our attention to this case.

Remark 2.2. *More generally we could imagine a model of overconfidence where* $\lambda_A(N,t) = A_A + B_A\hat{\lambda}(N,t)$ *and* $\lambda_B(N,t) = A_B + B_B\hat{\lambda}(N,t)$ *and* $\hat{\lambda}(N,t) = \frac{\hat{z}+N}{\hat{k}+t}$. *This model has simple closed form expressions for the Harrison Kreps minimal consistent equilibrium prices.*

Figure 1 plots the upper bound for the Harrison Kreps equilibrium minimal consistent prices in Equation (2.53) and a lower bound which is the private valuation for investor class B. These bounds are shown for $N = 0$, $N = 1$, and $N = 2$ versus time. Figure 2 shows the upper bound in Equation (2.54) on Harrison Kreps minimal consistent prices in state 0 and a lower bound which is the private valuation for investor class A. The values of $\lambda_A(0,0) = 0.6$ and $\lambda_B(0,0) = 0.5$ are chosen for comparison to the cases in Table 1. The figures show that the speculative premia are lower than in the case with no learning and diminish rapidly through time. In particular, the speculative premium in State 1 for Investor B for $N = 0$ with learning must be less than 0.311 while in the case with no learning in Table 1 the speculative premium in the corresponding no learning case it is 0.38. In state 0 the corresponding speculative premium for Investor A can be no larger than 0.32 while the corresponding speculative premium in the no learning case is 0.38. As time progresses, we see the speculative premia diminish rapidly.

Figure 3 displays the percentage difference between the upper bound for the Harrison Kreps equilibrium minimal consistent pricesin state 1 in Equation (2.53) and a lower bound which is the private valuation for investor class B. These bounds are shown for $N = 0$, $N = 1$, and $N = 2$ versus time. For reference the case with no learning is also displayed. Figure 4 shows the percentage difference between the upper

[5] The restriction in the proposition ensures the state 0 prices in the different economies decrease in t and increase in N. However, the restriction $z_A - z_B \leq (1 + k_A)\frac{\gamma k_B}{1-\gamma k_B}$ or $\gamma k_B > 1$ is also sufficient. If this restriction holds and the restriction in the proposition does not, then the prices in the different economies in state 0 can increase through time and decrease in N. This provides an illustration of the Keynsian beauty contest; speculators value the asset for the resale.

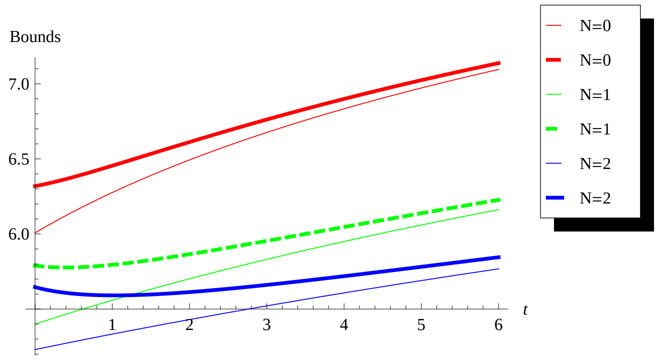

Figure 1: Bounds vs. Time

This figure shows the upper bounds in Equation (2.53) on the Harrison Kreps equilibrium stock price in state 1 and the private valuation for investor class B versus time for $N = 0$, $N = 1$, and $N = 2$. Parameters: $\lambda_A(0,0) = 0.6$, $\lambda_B(0,0) = 0.5$, $z_A = z_B = 1$, and $\gamma = 0.1$.

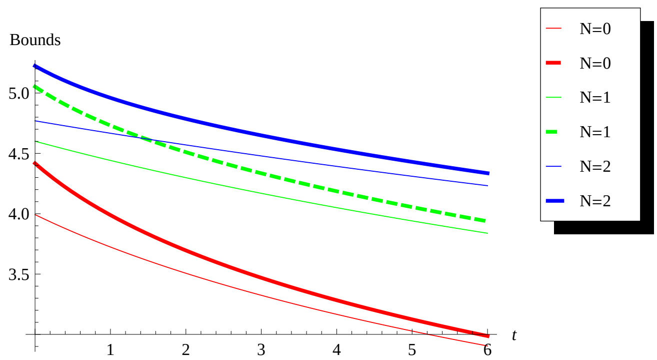

Figure 2: Bounds vs. Time

This figure shows the upper bounds in Equation (2.54) on the Harrison Kreps equilibrium stock price in state 0 and the private valuation for investor class A versus time for $N = 0$, $N = 1$, and $N = 2$. Parameters: $\lambda_A(0,0) = 0.6$, $\lambda_B(0,0) = 0.5$, $z_A = z_B = 1$, and $\gamma = 0.1$.

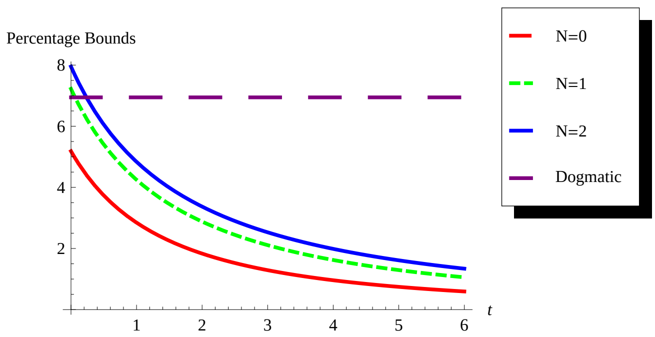

Figure 3: Percentage Speculative Premium Bounds vs. Time
This figure shows the percentage difference between the upper bounds in Equation (2.53) on the Harrison Kreps equilibrium stock price in state 1 and the private valuation for investor class B versus time for $N = 0$, $N = 1$, and $N = 2$. The horizontal line represents the case with no learning. Parameters: $\lambda_A(0, 0) = 0.6$, $\lambda_B(0, 0) = 0.5$, $z_A = z_B = 1$, and $\gamma = 0.1$.

bound in Equation (2.54) on Harrison Kreps minimal consistent prices in state 0 and a lower bound which is the private valuation for investor class A. Again the case with no learning is included. The values of $\lambda_A(0, 0) = 0.6$ and $\lambda_B(0, 0) = 0.5$ are chosen for comparison to the cases in Table 1. The figures show that the percentage speculative premia are lower than in the case with no learning and diminish rapidly through time. For example the upper bound on the percentage difference between Investor B's private valuation and the minimal consistent Harrison Kreps minimal consistent state 1 equilibrium price fo $N = 0$ is less than 5.2 % while in the no learning case it is approximately equal to 7 %. The corresponding numbers in state 0 for Investor A are 7.78 % and 8.3 %.

By taking limits in Equations (2.32) and (2.33) we can also deduce the following limits

$$\lim_{t\to\infty} S^Q(1, N, t) = \frac{1}{\gamma} \quad \lim_{t\to\infty} S^Q(0, N, t) = 0 \tag{2.58}$$

$$\lim_{N\to\infty} S^Q(\iota, N, t) = \lim_{N\to\infty} S^Q(1 - \iota, N, t) \tag{2.59}$$

Our final result in this section is to show that the Harrison Kreps minimal consistent prices converge to the true value.

Proposition 2.8. *When both investors learn, the minimal consistent Harrison Kreps equilibrium prices converge to the true values, that is almost surely (P^A or P^B) we*

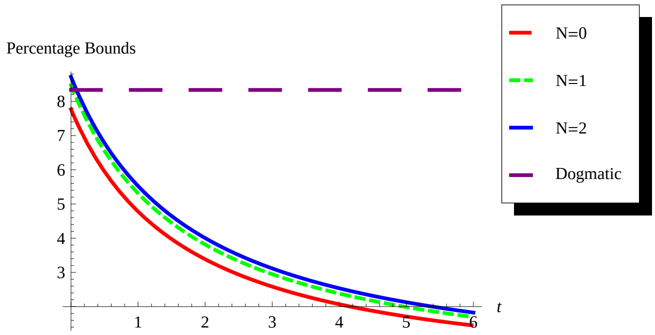

Figure 4: Percentage Speculative Premium Bounds vs. Time
This figure shows the percentage difference between the upper bounds in Equation (2.54) on the Harrison Kreps equilibrium stock price in state 0 and the private valuation for investor class A versus time for $N = 0$, $N = 1$, and $N = 2$. The horizontal line is the no learning case. Parameters: $\lambda_A(0,0) = 0.6$, $\lambda_B(0,0) = 0.5$, $z_A = z_B = 1$, and $\gamma = 0.1$.

have

$$\lim_{t\to\infty} S^Q(1, N_t, t) = \frac{\gamma + \lambda}{\gamma(\gamma + 2\lambda)} \tag{2.60}$$

$$\lim_{t\to\infty} S^Q(0, N_t, t) = \frac{\lambda}{\gamma(\gamma + 2\lambda)} \tag{2.61}$$

2.2 Analysis of Leverage

We now examine how not clearing the market for the riskless asset impacts the Harrison Kreps equilibrium. Aggregating each agents budget constraint and using the fact that we clear the stock market $\theta_t^A + \theta_t^B = 1$ gives

$$d(W_t^A + W_t^B) = \left\{\gamma\left(\alpha_t^A + \alpha_t^B\right)e^{\gamma t} + \delta_t - c_t^A - c_t^B\right\}dt + dS_t \tag{2.62}$$

Using the fact that $W_t^A + W_t^B - S_t = (\alpha_t^A + \alpha_t^B)e^{\gamma t}$ gives

$$d\left(\alpha_t^A + \alpha_t^B\right)e^{\gamma t} = \left\{\gamma\left(\alpha_t^A + \alpha_t^B\right)e^{\gamma t} + \delta_t - c_t^A - c_t^B\right\}dt \tag{2.63}$$

Which then gives

$$W_t^A + W_t^B - S_t = \left(\alpha_t^A + \alpha_t^B\right)e^{\gamma t} = e^{\gamma t}\int_0^t e^{-\gamma s}\left\{\delta_s - c_s^A - c_s^B\right\}ds \tag{2.64}$$

This expression is similar to the analysis in Loewenstein and Willard(2006) and links the present value of non-clearing in the consumption market to the lack of clearing in the riskless market, given by $W_t^A + W_t^B - S_t$.

There are three probability measures of interest,speculator A's probability measure, P^A, speculator B's probability measure, P^B, and the pricing measure, Q. While each speculator's expected discounted wealth satisfies the tranversality condition under their probability measure, it is not guaranteed to satisfy the tranversality condition under the other two measures. In the following propositions, we choose consumption strategies so that each speculator's expected discounted wealth satisfies the tranversality condition in their own measure as well as the Q measure.

Proposition 2.9. *In any Harrison Kreps equilibrium, when $\underline{\theta} = 0$, for any deterministic nonnegative c_t^A and c_t^B which satisfy*

$$\int_0^\infty e^{-\gamma t} c_t^i dt = \theta_i S(\iota, 0, 0) \tag{2.65}$$

the speculative premium for each investor is given by the expected present value of the excess consumption, that is for $i = A, B$

$$\begin{aligned} S(\iota,0,0) - S^i(\iota,0,0) &= E^i\left[\int_0^\infty e^{-\gamma s}\left\{c_s^A + c_s^B - \delta_s\right\} ds\right] \\ &= \lim_{t\to\infty} E^i\left[e^{-\gamma t}\left(S_t - W_t^A - W_t^B\right)\right] \\ &= E^i\left[\int_0^\infty e^{-\gamma s}\theta_s^j(\lambda_j(N_s,s) - \lambda_i(N_s,s))\Delta S_s ds\right] + \lim_{t\to\infty} E^i\left[e^{-\gamma t} S_t\right] \end{aligned} \tag{2.66}$$

In particular, when $\lambda_A(N,t) > \lambda_B(N,t)$ for all N and t, investor i's expected present value of investor j's trading losses are financed from violating the transverality condition under P^i,

$$\lim_{t\to\infty} E^i\left[e^{-\gamma t} W_t^j\right] = E^i\left[\int_0^\infty e^{-\gamma s}\theta_s^j(\lambda_i(N_s,s) - \lambda_j(N_s,s))\Delta S_s ds\right] < 0. \tag{2.67}$$

These findings say, restricting attention to deterministic consumption choices, any speculative premium for investor i is the expected present value of the non-clearing in the consumption market which is exactly equal to the asymptotic expected present value of the amount by which the riskless market does not clear by. Furthermore, using the tranversality condition on W^i, this asymptotic non-clearing in the riskless asset market can be written

$$\lim_{t\to\infty} E^i[e^{-\gamma t} S_t] - \lim_{t\to\infty} E^i[e^{-\gamma t} W_t^j] \tag{2.68}$$

which attributes nonclearing in the riskless market to 1) a possible violation of the law of one price and 2) investor j's expected trading losses.

When the stock price is the minimal consistent price the speculative premium is generated by the non-clearing in the consumption market which is in turn investors i's expectation of investor j's trading losses. Moreover, from each investor's perspective, they expect the other investor's wealth to grow unboundedly negative to finance their expected trading losses.

When arbitrary budget feasible optimal consumption choices are allowed, then the equality in Equation (2.67) may not hold and the speculative premium is given by

$$
\begin{aligned}
S(\iota,0,0) - S^i(\iota,0,0) = E^i & \left[\int_0^\infty e^{-\gamma s} \left\{ c_s^A + c_s^B - \delta_s \right\} ds \right] \\
& + \lim_{t\to\infty} E^i \left[e^{-\gamma t} W_t^j \right] - E^i \left[\int_0^\infty e^{-\gamma s} \theta_s^j (\lambda_i(N_s,s) - \lambda_j(N_s,s)) \Delta S_s ds \right].
\end{aligned}
$$

Again, the speculative premium is driven by non-clearing; in this case it contains an additional term to reflect asymptotic non-clearing of the riskless market less the amount required to finance expected trading losses.

Interestingly, any asset pricing bubble (a violation of the law of one price) is supported entirely by the riskless market. The next proposition deals with violations of the law of one price. A similar observation is in Loewenstein and Willard(2006).

Proposition 2.10. *In any Harrison Kreps equilibrium, when $\underline{\theta} = 0$, for any deterministic, nonnegative c_t^A and c_t^B which satisfy*

$$
\int_0^\infty e^{-\gamma t} c_t^i dt = \theta_i S(\iota,0,0) \tag{2.69}
$$

under the Q measure we have

$$
\begin{aligned}
S(\iota,0,0) - S^Q(\iota,0,0) = E^Q & \left[\int_0^\infty e^{-\gamma s} \left\{ c_s^A + c_s^B - \delta_s \right\} ds \right] \\
& = \lim_{t\to\infty} E^Q \left[e^{-\gamma t} \left(S_t - W_t^A - W_t^B \right) \right] \qquad (2.70)
\end{aligned}
$$

where S^Q is the minimal consistent price. The bubble component is driven entirely by non-clearing in the consumption and riskfree markets.

While one might think that the transversality conditions would rule out an asset pricing bubble, this is not the case. In fact, our restriction to deterministic consumption plans ensures the investorss' wealth processes satisfy the transversality conditions under the Q measure. However, the non-clearing in the riskless asset market does not satisy these conditions. Because investors have a source of wealth from the asymptotic non-clearing at the horizon, they can consume more.

Proposition 2.10 says a violation of the law of one price can be supported by lack of clearing in the consumption market. This can occur even if speculators share a common prior, but would have to be suppported by infinite retrade. In fact, ruling out this possibility by insisting markets clear would potentially rule out the possibility of any speculative premium at all. These findings suggest the Harrison Kreps equilibrium might be appropriate for understanding very small isolated markets but for explaining general market conditions, one would require an explanation of the resources outside the model. This suggests we examine these economies from other perspectives.

3 Speculation with Limited Capital

A funny feature of the above model is that speculators can lose a lot. The optimal solution in equilibrium will involve negative wealth, in which case speculators may choose to optimally default. Moreover, the equilibrium prices above are the same regardless of whether there are no short sales involved or if a large but finite amount of short sales are permitted. In this section we examine an alternative model in which markets clear and speculators are optimally not defaulting. We begin with the choice problem. We do not prohibit short sales explicitly. However, we explicitly constrain wealth to be nonnegative which will implicitly limit short sales. We show in a later section even if investors are allowed to have some negative wealth (and hence be able to walk away from their debts), they will optimally choose not to do so in equilibrium. The restriction to nonnegative wealth can be thought of as a margin constraint which perfectly anticipates price changes.

Once again we postulate candiate equilibrium prices only now we allow for a possibly stochastic locally riskless interest rate which is a function of N_t and t; in other words $r_t \equiv r(N_t, t)$. We suppose the stock price is given by

$$dS_t = (r_t S_{t-} - \delta_t - \lambda_t^Q \Delta S_t)dt + \Delta S_t dN_t \tag{3.71}$$

The equity premium for individual i is given by $\frac{(\lambda_i(N_t,t) - \lambda_t^Q)\Delta S_t}{S_{t-}}$ which can be positive, negative or 0.

Let $R_t = e^{\int_0^t r_s ds}$.

Choice Problem 3.1 (Choice Problem with Limited Resources)**.** *Given securities endowments θ^i, choose predictable securities holdings θ_t and α_t and adapted consumption c_t to maximize*

$$E^i \left[\int_0^\infty e^{-\gamma t} c_t dt \right]$$

subject to

$$dW_t^i = \alpha_t dR_t + \theta_t dS_t + \theta_t \delta_t dt - c_t dt$$

and

$$W_t \geq 0 \Rightarrow \theta_t \Delta S_t \geq -W_{t-}$$

and $c_t \geq 0$ where $W_t = \alpha_t R_t + \theta_t S_t$ and $W_0 = \theta^i S_0$.

Proposition 3.1. *The value function is given by*

$$V^i(W, N, t) = W h^i(N, t) \tag{3.72}$$

The function $h^i(N, t) \geq 1$ satisfies

$$\frac{\partial h^i(N, t)}{\partial t} + h^i(N, t)(r_t - \gamma - \lambda_i(N, t)) + \lambda^Q(N, t) h^i(N, t) = 0 \tag{3.73}$$

Moreover,

$$\lambda_i(N, t) h^i(N + 1, t) \leq \lambda^Q(N, t) h^i(N, t) \tag{3.74}$$

with equality when $\theta_t \Delta S_t > -W_t$. The optimal consumption satisfies

$$c_t(1 - h^i(N, t)) = 0 \tag{3.75}$$

and when $h^i(N_t, t) = 1$, $r_t + \lambda^Q(N, t) = \gamma + \lambda_i(N_t, t)$, $\ell \times P^i$ almost surely.

Corollary 3.1. *If an optimal solution does not involve default we have $e^{-\gamma t} h^i(N_t, t) S_t + \int_0^t e^{-\gamma s} h^i(N_s, s) \delta_s ds$ and $e^{\int_0^t (r_s - \gamma) ds} h^i(N_t, t)$ are P^i martingales. Therefore if $\Delta S_t \neq 0$ λ^Q is unique and we have*

$$\frac{e^{\int_0^t (r_s - \gamma) ds} h^i(N_t, t)}{h^i(0, 0)} = \frac{dQ}{dP^i}|_t \equiv M_t^i \tag{3.76}$$

Moreover, if $r_t \leq \gamma$, $h^i(N_t, t)$ is a P^i submartingale and if $r_t = \gamma$, $h^i(N_t, t)$ is a P^i martingale. Therefore, if $r_t = \gamma\ \forall t \geq \tau$, if $h^i(N_\tau, \tau) = 1$ then $h^i(N_t, t) = 1$ for all $t \geq \tau$. Moreover, we have

$$\begin{aligned} W^i(0) = \frac{1}{h^i(0,0)} E^i \left[\int_0^\infty e^{-\gamma t} c_t 1_{\{h^i(N_t,t)=1\}} dt \right] \\ = E^Q \left[\int_0^\infty e^{-\int_0^t r_s ds} c_t 1_{\{h^i(N_t,t)=1\}} dt \right] \\ = E^i \left[\int_0^\infty e^{-\int_0^t r_s ds} M_t^i c_t 1_{\{h^i(N_t,t)=1\}} dt \right] \end{aligned} \quad (3.77)$$

The function h^i impounds the future speculative opportunities. It represents the value of one unit of wealth optimally invested, or the marginal utility of wealth. In an i.i.d. framework, $h^i = 1$ or there is no optimal solution. However, it is important to emphasize that when $r_t \leq \gamma$, then $h^i(N_t, t)$ is a submartingale. In effect, the submartingale property lowers the discount rate on future wealth to reflect future speculative opportunities. This is an important observation in our next section.

Remark 3.1. *The optimal policy involves local indeterminancy in the portfolio and consumption choice. It is important to check the transversality condition*

$$\lim_{t \to \infty} E^i[e^{-\gamma t} h^i(N_t, t) W_t] = 0 \quad (3.78)$$

which follows from (3.77).

3.1 Equilibrium

Our definition of equilibrium is standard.

Definition 3.1. *An equilibrium is an interest rate $r(N, t)$ and stock price $S(\iota, N, t)$ such that agents solve choice problem 3.1 and the consumption market, the riskless asset market, and the stock market all clear, that is $\theta_t^A + \theta_t^B = 1$, $c_t^A + c_t^B = \delta_t$, and $\alpha_t^A + \alpha_t^B = 0$, $\ell \otimes P^i$ almost surely where c_t^i, θ_t^i and α_t^i solve problem 3.1.*

Many models of speculation assume either infinitely wealthy investors or unlimited borrowing at a fixed riskless rate. In addition, these models either use exponential utility or risk neutral preferences. These assumptions rule out wealth effects and this implies any individual can be the marginal buyer of the stock. However, if speculators face lower bounds on wealth, even these preferences will give rise to wealth effects. The next section examines equilibrium restrictions which are only implied by market clearing.

3.1.1 Leverage and Collateral

In this section we explore equilibrium limits and speculation and leverage given market clearing.

Proposition 3.2. *In any equilibrium where markets clear,* $0 \leq W^A \leq S^A$ *and* $0 \leq W^B \leq S^B$*. As a result,*

$$W^i_{t-} + \theta^i_t \Delta S_t \geq S_t - S^j_t \tag{3.79}$$

In particular, if $W^j_{t-} > S^j_t$ *and* $\Delta S_t < 0$*, then* j *is the marginal buyer of the stock.*

The importance of this result is that it relies only on market clearing. Equation (3.79) indicates that if there is a speculative premium when the reqime shifts, then the wealth constraint for agent i cannot bind for any agent. In fact, margin requirements can be tighter than simply $\theta \Delta S > -W_{t-}$ whenever a speculative premium persists. This proposition also indicates that when agent i is not wealthy and $\Delta S < 0$, then investor j must be the marginal buyer of the stock. This contrasts with the Harrison Kreps equilibrium where the marginal buyer is alway the buyer with the highest valuation.

3.1.2 Default

Although our investors have finite marginal utility for zero consumption, they will not default in equilibrium if their disagreements are persistent enough. This is because if they default, then the remaining investors set prices. Given these prices, an investor with different beliefs can enjoy very large utility. Therefore rather than default, an investor will optimally set a tiny amount of capital to speculate at very favorable prices. But this then says default cannot be part of the equilibrium. This is suprising because this is true even when investors learn and after an infinite amount of data both learn the true parameter λ. One might imagine that the gains from staying solvent would diminish through time. The next proposition gives the precise result.

Proposition 3.3. *Suppose $k_B \geq k_A$ and $z_B < z_A$ (so that $\lambda^A(N,t) > \lambda^B(N,t)$ for all N and t) or investors have dogmatic beliefs. In equilibrium, the wealth constraint does not bind for either investor.*

Remark 3.2. *In the case $z_A = z_B$ and $k_A < k_B$, the proof of Proposition 3.3 still implies investor A will have infinite expected utility when facing prices set by investor B. However, the proof does not imply investor class B will have infinite expected utility when facing prices set by investor A. Nevertheless, our construction of the equilibrium indicates in this case investor class B will still not have a binding wealth constraint; the value of staying alive exceeds any possible benefit of taking a highly leveraged position which leads to default in some states.*

To get an equilibrium with potential default, one could assume investors have a finite, deterministic horizon. However, in this case prices will be lower. Any retrade value will vanish as the horizon approaches and investors might optimally choose to default.

It is also worth noting that this same propostion can be applied in various other settings.

3.1.3 Equilibrium Prices

We now characterize equilibrium in our economy. We first describe an important stochastic process for describing the equilibrium is the change of measure from P^A to P^B restricted to time t. This is the random variable Z_t such that for any $\mathcal{F}_t$ measurable random variable x, $E^B[x] = E^A[Z_t x]$. In the case where investors learn this is given by

$$Z_t = \frac{\frac{\Gamma(N_t+z_B)}{\Gamma(z_B)} \frac{k_B^{z_B}}{(t+k_B)^{N_t+z_B}}}{\frac{\Gamma(N_t+z_A)}{\Gamma(z_A)} \frac{k_A^{z_A}}{(t+k_A)^{N_t+z_A}}} \tag{3.80}$$

and when investors have dogmatic beliefs

$$Z_t = e^{(\lambda_A - \lambda_B)t} \left(\frac{\lambda_B}{\lambda_A} \right)^{N_t} \tag{3.81}$$

In both cases Z_t satisfies the stochastic differential equation

$$dZ_t = \frac{\lambda_B(N_{t-},t) - \lambda_A(N_{t-},t)}{\lambda_A(N_{t-},t)} Z_{t-} \left(dN_t - \lambda_A(N_{t-},t)dt \right) \quad Z_0 = 1 \tag{3.82}$$

Our first result derives the state price density and interest rate for the equilibrium.

Proposition 3.4. *Suppose $\lambda_A(N,t) > \lambda_B(N,t)$ for all N and t. The state price density for investor A is given by*

$$\rho_t^A = e^{-\gamma t}\frac{\max\left(\eta,(1-\eta)Z_t\right)}{\max(\eta,1-\eta)} \tag{3.83}$$

where η is the solution to

$$\theta^A E^A\left[\int_0^\infty \rho_t^A\delta_t dt\right] = E^A\left[\int_0^\infty \rho_t^A\delta_t 1_{\{\eta>(1-\eta)Z_t\}}dt\right]. \tag{3.84}$$

The equilibrium interest rate is given by

$$r_t = \begin{cases} \gamma \quad \textit{If} \quad \eta > (1-\eta)Z_{t-} \\ \gamma + \lambda_B(N_t,t) - \frac{\eta\lambda_A(N_t,t)}{(1-\eta)Z_t} \quad \textit{If} \quad (1-\eta)Z_{t-} > \eta \quad \textit{and} \quad \frac{\lambda_B(N_t,t)}{\lambda_A(N_t,t)}(1-\eta)Z_{t-} < \eta \\ \gamma \quad \textit{If} \quad (1-\eta)Z_{t-} > \eta \quad \textit{and} \quad \frac{\lambda_B(N_t,t)}{\lambda_A(N_t,t)}(1-\eta)Z_{t-} > \eta \end{cases} \tag{3.85}$$

The equilibrium λ^Q is given by

$$\lambda_t^Q = \begin{cases} \lambda_A(N_t,t) \quad \textit{If} \quad \eta > (1-\eta)Z_{t-} \\ \frac{\eta\lambda_A(N_t,t)}{(1-\eta)Z_t} \quad \textit{If} \quad (1-\eta)Z_{t-} > \eta \quad \textit{and} \quad \frac{\lambda_B(N_t,t)}{\lambda_A(N_t,t)}(1-\eta)Z_{t-} < \eta \\ \lambda_B(N_t,t) \quad \textit{If} \quad (1-\eta)Z_{t-} > \eta \quad \textit{and} \quad \frac{\lambda_B(N_t,t)}{\lambda_A(N_t,t)}(1-\eta)Z_{t-} > \eta \end{cases} \tag{3.86}$$

In contrast to the Harrison Kreps model where the time stock prices can be thought of an investor's private valuation plus the option to resell the asset, in this setting at time t when $h^i(N,t) = 1$, future consumption is priced as investor i's private valuation of the consumption plus the option to resell consumption. Notice the riskfree rate satisfies the bounds $\gamma + \lambda_B(N_t,t) - \lambda_A(N_t,t) \leq r_t \leq \gamma$. The price of risk satisfies $\lambda_B(N_t,t) \leq \lambda_t^Q \leq \lambda_A(N_t,t)$.

There are three important regions for understanding equilibrium pricing. When $\eta > (1-\eta)Z_t$, $\lambda_t^Q = \lambda_A(N_t,t)$ and $r_t = \gamma$. In this region, there is no equity premium for investor class A, but investor class B has a risk premium of $(\lambda_B(N_t,t) - \lambda_A(N_t,t))\frac{\Delta S}{S}$. This is positive when $\Delta S < 0$ and negative when $\Delta S > 0$. In this region investor class A consumes the dividend, while investor class B saves and builds capital.

The next region is when $(1-\eta)Z_t > \eta > (1-\eta)Z_t\frac{\lambda_B(N_t,t)}{\lambda_A(N_t,t)}$. In this region, $r_t < \gamma$ and both investors have a non-zero equity premium. Investor class B consumes the dividend and investor class A saves.

The final region is when $(1-\eta)Z_t\frac{\lambda_B(N_t,t)}{\lambda_A(N_t,t)}$. In this region, $r_t = \gamma$ and investor A has a non-zero equity premium while investor B does not. In this region investor class B consumes the dividend and investor class A saves.

Intuitively, these regions correspond when A dominates the wealth distribution, a transition region, and when B dominates the wealth disribution. If the economy begins in the first region, then as time passes this tends to favor investor class B. If the state doesn't change then the economy will transition through the three regions. However, changes in state lower $(1-\eta)Z_t$ so state transitions will tend to bring the economy back to region 1. State transitions favor investor class A so state transitions tend to result in A dominating the wealth distribution.

The next proposition summarizes asset prices when all markets clear.

Proposition 3.5. *Suppose $\lambda_A(N_t,t) > \lambda_B(N_t,t)$. Equilibrium prices are given by*

$$S(1,N_t,t) = \frac{\eta}{\max(\eta,(1-\eta)Z_t)} \sum_{j=0}^{\infty} \int_0^{t^*(2j,Z_t,t)} e^{-\gamma s} P^A\{N_{s+t} = N_t + 2j | N_t\} ds + \frac{(1-\eta)Z_t}{\max(\eta,(1-\eta)Z_t)} \sum_{j=0}^{\infty} \int_{t^*(2j,Z_t,t)}^{\infty} e^{-\gamma s} P^B\{N_{s+t} = N_t + 2j | N_t\} ds \quad (3.87)$$

$$S(0,N_t,t) = \frac{\eta}{\max(\eta,(1-\eta)Z_t)} \sum_{j=0}^{\infty} \int_0^{t^*(2j+1,Z_t,t)} e^{-\gamma s} P^A\{N_{s+t} = N_t + 2j+1 | N_t\} ds + \frac{(1-\eta)Z_t}{\max(\eta,(1-\eta)Z_t)} \sum_{j=0}^{\infty} \int_{t^*(2j+1,Z_t,t)}^{\infty} e^{-\gamma s} P^B\{N_{s+t} = N_t + 2j+1 | N_t\} ds \quad (3.88)$$

where $t^*(j,Z,t)$ is the solution to

$$\eta P^A\{N_{t^*(j,Z,t)+t} = N_t + j | N_t\} = (1-\eta) Z P^B\{N_{t^*(j,Z,t)+t} = N_t + j | N_t\} \quad (3.89)$$

if a nonnegative solution exists and zero otherwise.

The next proposition gives the equilibrium wealth processes.

Proposition 3.6. *The equilibrium wealth process for investor class B is given by*

$$W^B(1,N_t,t) = \frac{(1-\eta)Z_t}{\max(\eta,(1-\eta)Z_t)} \sum_{j=0}^{\infty} \int_{t^*(2j,Z_t,t)}^{\infty} e^{-\gamma s} P^B\{N_{s+t} = N_t + 2j | N_t\} ds \quad (3.90)$$

$$W^B(0,N_t,t) = \frac{(1-\eta)Z_t}{\max(\eta,(1-\eta)Z_t)} \sum_{j=0}^{\infty} \int_{t^*(2j+1,Z_t,t)}^{\infty} e^{-\gamma s} P^B\{N_{s+t} = N_t + 2j | N_t\} ds \quad (3.91)$$

The equilibrium wealth process for investor class A can be obtained from market clearing:

$$W^A(\cdot,N,t) = S(\cdot,N,t) - W^B(\cdot,N,t) \quad (3.92)$$

When investors learn from the data, this affects portfolio choice and, in turn, equilibrium stock price dynamics. It is convenient to define the functions

$$f^A(1,N_t,t) = \sum_{j=0}^{\infty} \int_0^{t^*(2j,Z_t,t)} e^{-\gamma s} \left(\lambda^A(N_t,t) - \lambda^A(N_t+2j,s+t)\right) P^A\{N_{s+t} = N_t{+}2j|N_t\} ds \tag{3.93}$$

$$f^A(0,N_t,t) = \sum_{j=0}^{\infty} \int_0^{t^*(2j+1,Z_t,t)} e^{-\gamma s} \left(\lambda^A(N_t,t) - \lambda^A(N_t+2j+1,s+t)\right) P^A\{N_{s+t} = N_t{+}2j{+}1|N_t\} ds \tag{3.94}$$

$$f^B(1,N_t,t) = \sum_{j=0}^{\infty} \int_{t^*(2j,Z_t,t)}^{\infty} e^{-\gamma s} \left(\lambda^B(N_t,t) - \lambda^B(N_t+2j,s+t)\right) P^B\{N_{s+t} = N_t{+}2j|N_t\} ds \tag{3.95}$$

$$f^B(0,N_t,t) = \sum_{j=0}^{\infty} \int_{t^*(2j+1,Z_t,t)}^{\infty} e^{-\gamma s} \left(\lambda^B(N_t,t) - \lambda^B(N_t+2j+1,s+t)\right) P^B\{N_{s+t} = N_t{+}2j{+}1|N_t\} ds \tag{3.96}$$

These functions are identically 0 when investors have dogmatic beliefs. However, when investors learn from the data they play a role in the equilibrium dynamics.

Corollary 3.2.

$$\frac{\partial S(\cdot,N,t)}{\partial \eta} = \begin{cases} -\frac{1}{\eta(1-\eta)Z}\frac{W^B(\cdot,N,t)}{\lambda_B(N,t)-\lambda_A(N,t)} & \text{If } \; \eta > (1-\eta)Z \\ \frac{1}{\eta(1-\eta)Z}\frac{W^A(\cdot,N,t)}{\lambda_B(N,t)-\lambda_A(N,t)} & \text{If } \; (1-\eta)Z > \eta \end{cases} \tag{3.97}$$

As a function of η, stock price is highest when $\eta = (1-\eta)Z$. In addition, when $\eta \uparrow 1$ the stock price goes to A's private valuation and as $\eta \downarrow 0$ the stock price goes to B's private valuation.

$$\begin{aligned} &\frac{\partial S(\cdot,N,t)}{\partial t} \\ &= (\lambda_A(N,t)-\lambda_B(N,t))W^B(\cdot,N,t)1_{\{(1-\eta)Z_t<\eta\}} + (\lambda_B(N,t)-\lambda_A(N,t))W^A(\cdot,N,t)1_{\{(1-\eta)Z_t>\eta\}} \\ &\qquad + \frac{\eta}{\max(\eta,(1-\eta)Z_t)} f^A(\cdot,N,t) + \frac{(1-\eta)Z_t}{\max(\eta,(1-\eta)Z_t)} f^B(\cdot,N,t) \\ &= \left(r_t + \lambda_t^Q - \gamma - \lambda_A(N,t)\frac{W^A(\cdot,N,t)}{S(\cdot,N,t)} - \lambda_B(N,t)\frac{W^B(\cdot,N,t)}{S(\cdot,N,t)}\right) S(\cdot,N,t) \\ &\qquad + \frac{\eta}{\max(\eta,(1-\eta)Z_t)} f^A(\cdot,N,t) + \frac{(1-\eta)Z_t}{\max(\eta,(1-\eta)Z_t)} f^B(\cdot,N,t) \end{aligned} \tag{3.98}$$

Corollary 3.2 indicates the stock price will unambiguously increase in η when $\eta > (1-\eta)Z_t$ and decrease in η when $(1-\eta)Z_t > \eta$. Thus, the maximal stock price as a function of η is when $\eta = (1-\eta)Z_t$ although the stock fails to be differentiable at this point. The maximal price is above both investors' private valuations as in the Harrison Kreps model. However, it is possible for the price to be below an investor's private valuation if the investor has low wealth.

Figure 5 shows the equilibrium stock price versus time when investors have dogmatic beliefs. The upper line shows how the stock price moves in state 1 while the bottom line shows the resulting stock price when the state changes from one to zero. Notice the magnitude of the price change is the difference between the two lines. The magnitude of the changes grows until the peak, then shrinks, and then grows again. This will be examined in more detail later, but this behavior is important to understand how leverage and prices are linked. Figure 6 shows leverage in the two regimes. Prior to the peak, as investor class B builds capital he can take more leverage in state 1 despite the increase in volatility. However, in State 1 leverage peaks well after the peak prices. This is because the volatility shrinks after the peak prices; lowered volatility allows bigger leverage.

Figure 7 shows the stock price versus time but for dogmatic beliefs and a larger disagreement. Here we see the stock price volatility can hit zero and even move so that when the dividend drops the stock price goes up. As predicted from Corollary 3.2 the rate of change is higher when investors have larger disagreements. Figure 8 shows leverage in state 1. When the stock price volatility vanishes, the the speculators take infinite positions. This is similar to the non-existance example in Hart(1975) but here because this only occurs on a zero measure set the stochastic integrals describing trading gains are defined. It is a bit of an abstraction to say equilibrium exists here, however.

Figure 9 shows the stock price as a function of time when we start in state 1 and investors learn. The top line is the stock price in state 1 and the bottom line illustrates the stock price when the state shifts to state 0. Stock price jumps are increasing as time passes. Here there are two effects: 1) as time passes, investors tend to update their beliefs and 2) their beliefs tend to agree more. Figure 10 shows leverage versus time in state 1. Not suprisingly, leverage decreases as disagreement decreases.

Remark 3.3. *We can easily generalize this to the case where the stock pays a dividend*

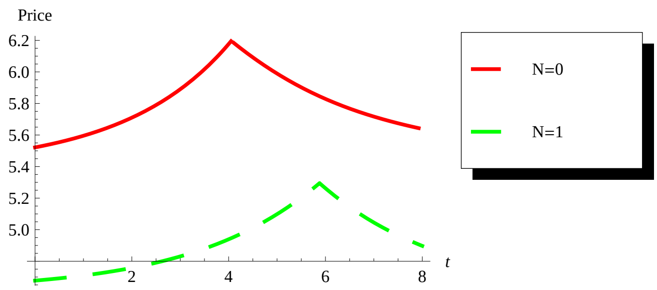

Figure 5: Price vs. Time

This figure shows the equilibrium stock price versus time for $N = 0$ and $N = 1$ when investors do not learn. Parameters: $\lambda_A(N,t) = 0.6$, $\lambda_B(N,t) = 0.5$, $\gamma = 0.1$.

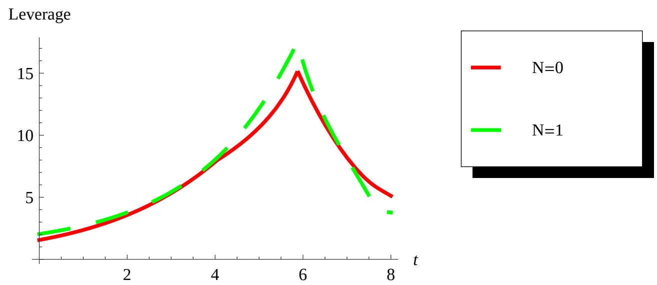

Figure 6: Leverage vs.Time

Parameters: $\lambda_A(N,t) = 0.6$, $\lambda_B(N,t) = 0.5$, $\gamma = 0.1$.

$\delta_t(1) = H$ and $\delta_t(0) = L$. The stock price in state 1 will be

$$HS(1,N,t) + LS(0,N,t) \tag{3.99}$$

and in state 0 the stock price will be

$$LS(1,N,t) + HS(0,N,t) \tag{3.100}$$

Figures 11, 12, 13, and 14 show how asset prices in both states at time 0 behave as a function of η for small disagreement and larger disagreement. When η is large, Investor class A dominates the wealth distribution and prices reflect A's private valuation. When η goes to $\frac{1}{2}$ prices go up to the maximal value which greatly exceeds each individuals private valuation and when η gets small, prices reflect Investor class B's private valuation. Loosely speaking, as more time is spent in each state, prices will tend to move in the direction of η getting smaller since Investor class A always

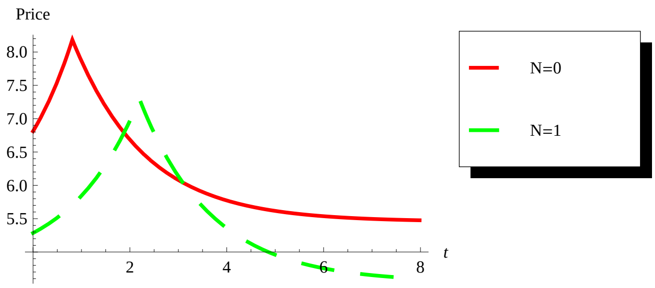

Figure 7: Price vs. Time

This figure shows the equilibrium stock price versus time for $N = 0$ and $N = 1$ when investors do not learn. Parameters: $\lambda_A(N,t) = 1.0$, $\lambda_B(N,t) = 0.5$, $\gamma = 0.1$.

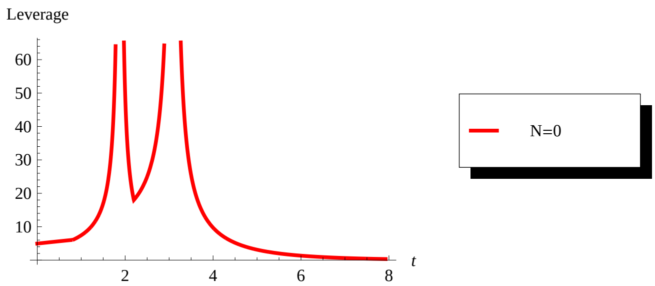

Figure 8: Leverage vs.Time

Parameters: $\lambda_A = 1.0$, $\lambda_B = 0.5$, $\gamma = 0.1$.

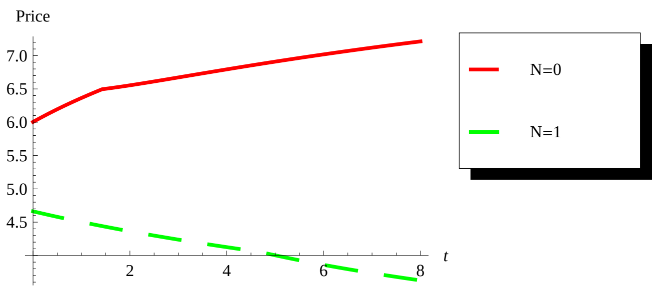

Figure 9: Stock Price versus Time.

This figure shows the stock price versus time with learning for $N = 0$ and $N = 1$. Parameters: $z_A = z_B = 1$, $\lambda_A(0,0) = 0.6$, $\lambda_B(0,0) = 0,5$, and $\gamma = 0.1$.

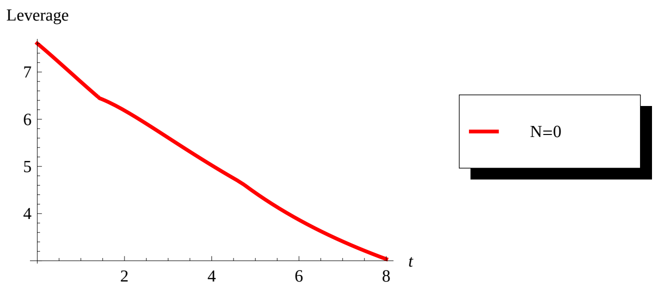

Figure 10: Leverage versus time in State 1.
This figure shows leverage versus time when investors learn for $N = 0$. Parameters: $z_A = z_B = 1$, $\lambda_A(0,0) = 0.6$, $\lambda_B(0,0) = 0,5$, and $\gamma = 0.1$.

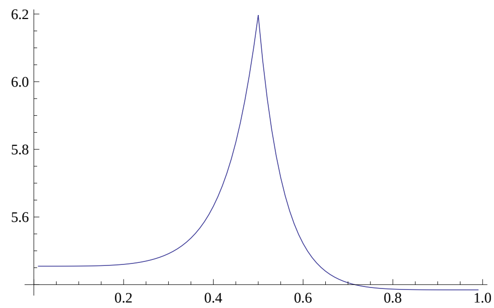

Figure 11: Stock Price in State 1 as a Function of η.
This figure shows how the stock price varies with η in state 1. Parameters: $\lambda_A = 0.6$, $\lambda_B = 0.5$, $\gamma = 0.1$.

estimates the state shifting faster than Investor class B, the more time spent in a given state tends to shift the wealth distribution in B's favor.

Figures 15, and 16 show how asset prices in both states at time 0 behave as a function of η for the case of learning. When η is large, Investor class A dominates the wealth distribution and prices reflect A's private valuation. When η goes to $\frac{1}{2}$ prices go up to the maximal value which greatly exceeds each individuals private valuation and when η gets small, prices reflect Investor class B's private valuation. Loosely speaking, as more time is spent in each state, prices will tend to move in the direction of η getting smaller since Investor class A always estimates the state shifting faster than Investor class B, the more time spent in a given state tends to shift the wealth distribution in B's favor.

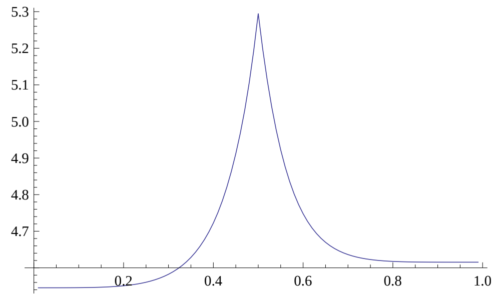

Figure 12: Stock Price in State 0 as a Function of η.
This figure shows how the stock price varies with η in state 0. Parameters: $\lambda_A = 0.6$, $\lambda_B = 0.5$, $\gamma = 0.1$.

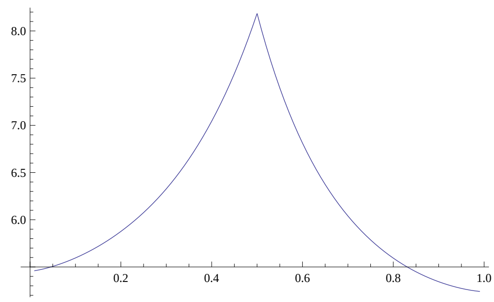

Figure 13: Stock Price in State 1 as a Function of η.
This figure shows how the stock price varies with η in state 1 when agents have larger disagreement. Parameters: $\lambda_A = 1.0$, $\lambda_B = 0.5$, $\gamma = 0.1$.

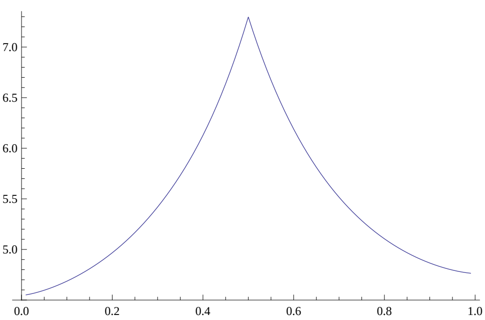

Figure 14: Stock Price in State 0 as a Function of η.
This figure shows how the stock price varies with η in state 1 when agents have larger disagreement. Parameters: $\lambda_A = 1.0$, $\lambda_B = 0.5$, $\gamma = 0.1$.

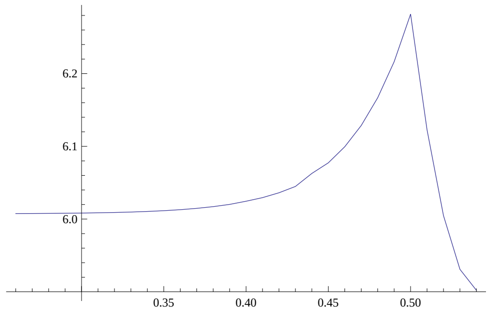

Figure 15: Stock Price in State 1 versus η.
This figure shows stock price in state 1 for $N = 0$ versus η. Parameters: $z_A = z_B = 1$, $\lambda_A(0,0) = 0.6$, $\lambda_B(0,0) = 0,5$, and $\gamma = 0.1$.

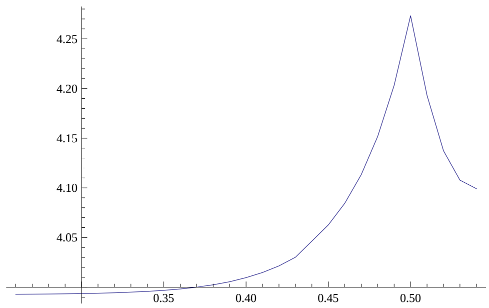

Figure 16: Stock Price in State 0 versus η.
This figure shows stock price in state 0 for $N = 0$ versus η. Parameters: $z_A = z_B = 1$, $\lambda_A(0,0) = 0.6$, $\lambda_B(0,0) = 0,5$, and $\gamma = 0.1$.

3.1.4 Equilibrium Stock Price Volatility

In this section, we look at the equilibrium stock price changes.

Proposition 3.7. *Equilibrium percentage stock price jumps are given by*

$$\frac{\Delta S(\cdot,N,t)}{S(\cdot,N,t)} \equiv \frac{S(1-\cdot,N+1,t)-S(\cdot,N,t)}{S(\cdot,N,t)}$$
$$= \left(\frac{\gamma+\lambda_A(N,t)}{\lambda_t^Q}-1\right)\frac{W^A(\cdot,N,t)}{S(\cdot,N,t)} + \left(\frac{\gamma+\lambda_B(N,t)}{\lambda_t^Q}-1\right)\frac{W^B(\cdot,N,t)}{S(\cdot,N,t)} - \frac{\delta(\cdot)}{\lambda_t^Q S(\cdot,N,t)}$$
$$- \frac{\eta}{\lambda_A(N,t)\max(\eta,(1-\eta)Z_t\frac{\lambda_B(N,t)}{\lambda_A(N,t)})}\frac{f^A(\cdot,N,t)}{S(\cdot,N,t)} - \frac{(1-\eta)Z_t}{\lambda_A(N,t)\max(\eta,(1-\eta)Z_t\frac{\lambda_B(N,t)}{\lambda_A(N,t)})}\frac{f^B(\cdot,N,t)}{S(\cdot,N,t)} \tag{3.101}$$

Proposition 3.7 indicates there are three distinct regions to analyze to understand the equilibrium stock price jumps. When $\eta > (1-\eta)Z$ percentage stock price changes unambiguously decrease as η decreases. When $(1-\eta)Z\frac{\lambda_B}{\lambda_A} > \eta$, percentage stock price changes also unambiguously decrease as η decreases. When $\eta \downarrow 0$, percentage stock price changes approach those of an economy populated by investors with B's beliefs and when $\eta \uparrow 1$ percentage stock price changes approach those of an economy populated by investors with A's beliefs. In the middle region, where $(1-\eta)Z\frac{\lambda_B}{\lambda_A} < \eta < (1-\eta)Z$, the percentage stock price changes increase as η decreases. This is caused by the fact that when the state changes in this region, the marginal buyer for consumption also changes.

Figures 17, 18, 19, and 20 show the stock price changes as a function of η for a moderate disagreement and a larger disagreement.

Figures 21 and 22 show the percentage change in stock price versus η in states 1 and 0 for $N = t = 0$ when investors learn. The pattterns are similar to the case where investors don't learn because the effects of learning show up when we vary N and t.

In general, we see that stock price volatility can be quite different depending on the equilibrium wealth distribution.

3.1.5 Analysis of Stock Price Changes with Dogmatic Beliefs

To get some insight into how the stock price changes behave we now analyze these changes in the three regions. In the first region when $\eta > (1-\eta)Z_t$ stock price jumps

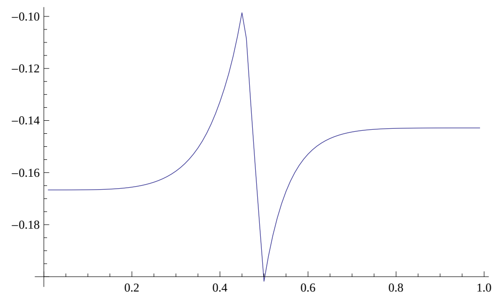

Figure 17: Stock Price Percentage Change in State 1 as a Function of η. This figure shows how $\frac{\Delta S}{S}$ varies with η in State 1 when investors have dogmatic beliefs. Parameters: $\lambda_A = 0.6$, $\lambda_B = 0.5$, $\gamma = 0.1$.

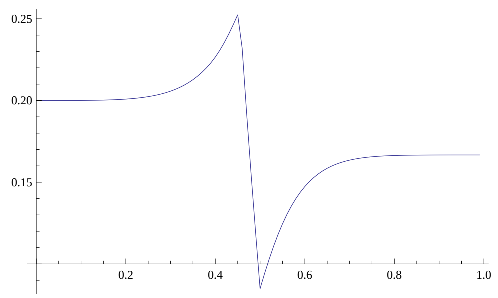

Figure 18: Stock Price Percentage Change in State 0 as a Function of η. This figure shows how $\frac{\Delta S}{S}$ varies with η in State 0 when investors have dogmatic beliefs. Parameters: $\lambda_A = 0.6$, $\lambda_B = 0.5$, $\gamma = 0.1$.

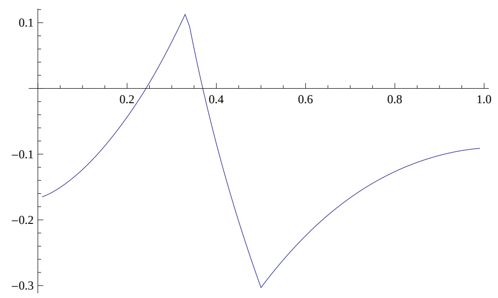

Figure 19: Stock Price Percentage Change in State 1 as a Function of η. This figure shows how $\frac{\Delta S}{S}$ varies with η in State 1 when agents have larger disagreement and dogmatic beliefs. Parameters: $\lambda_A = 1.0$, $\lambda_B = 0.5$, $\gamma = 0.1$.

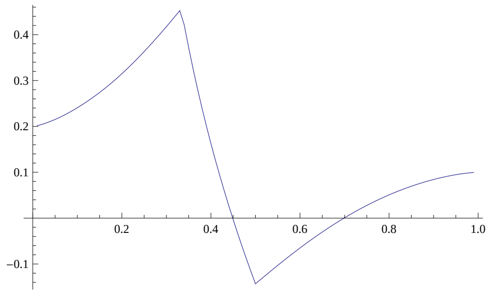

Figure 20: Stock Price Percentage Change in State 0 as a Function of η.
This figure shows how $\frac{\Delta S}{S}$ varies with η in State 0 when agents have larger disagreement and dogmatic beliefs. Parameters: $\lambda_A = 1.0$, $\lambda_B = 0.5$, $\gamma = 0.1$.

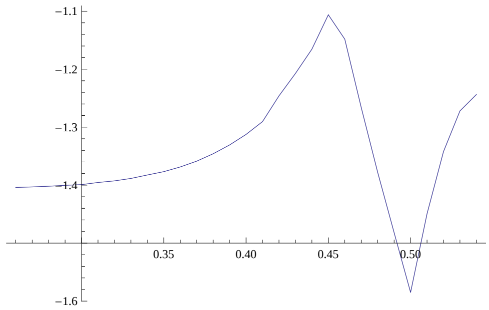

Figure 21: Percentage change in Stock Price in State 1 versus η.
This figure show the percentage change in stock price in state 1 when investors learn for $N = 0$ versus η. Parameters: $z_A = z_B = 1$, $\lambda_A(0,0) = 0.6$, $\lambda_B(0,0) = 0,5$, and $\gamma = 0.1$.

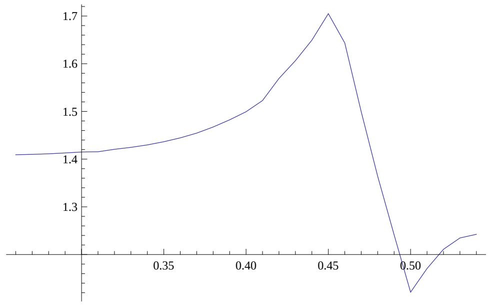

Figure 22: Percentage change in Stock Price in State 0 versus η.
This figure shows the percentage change in stock price in state 0 when investors learn for $N = 0$. Parameters: $z_A = z_B = 1$, $\lambda_A(0,0) = 0.6$, $\lambda_B(0,0) = 0,5$, and $\gamma = 0.1$.

are unambiguously negative in state 1. We have

$$\Delta S = \left(\frac{\gamma+\lambda_A}{\lambda_A}-1\right)W^A+\left(\frac{\gamma+\lambda_B}{\lambda_A}-1\right)W^B-\frac{1}{\lambda_A} < \left(\frac{\gamma+\lambda_A}{\lambda_A}-1\right)(W^A+W^B)-\frac{1}{\lambda_A}$$
$$= \frac{\gamma S}{\lambda_A}-\frac{1}{\lambda_A} < 0. \quad (3.102)$$

In addition we can see that the stock price jumps must become more negative as B's wealth increases, in other words as $\eta \to (1-\eta)Z_t$.

Howevever, large disagreements might cause stock price jumps to be positive when $\eta < (1-\eta)Z_t$. That is the dividend drops, the stock price goes up. This occurs due to an interest rate effect. When $(1-\eta)Z_t\frac{\lambda_B}{\lambda_A} > \eta$ we have

$$\Delta S = \left(\frac{\gamma+\lambda_A}{\lambda_B}-1\right)W^A+\left(\frac{\gamma+\lambda_B}{\lambda_B}-1\right)W^B-\frac{1}{\lambda_B}$$
$$= \frac{\gamma W^B-1}{\lambda_B}+\frac{\gamma+\lambda_A-\lambda_B}{\lambda_B}W^A \quad (3.103)$$

So if W^A is large, then the stock price jump can be positive. But as $\eta \to 0$, the stock price approaches B's private valuation so for small values of η, W^A is small and the stock price jump must be negative in state 1. For intermediate values the magnitude of the stock price jump can be positive or negative in state 1 depending on the magnitude of disagreement.

In state 0, stock price jumps are unambiguously positive when $(1-\eta)Z_t\frac{\lambda_B}{\lambda_A} > \eta$. We have

$$\Delta S = \left(\frac{\gamma+\lambda_A}{\lambda_B}-1\right)W^A+\left(\frac{\gamma+\lambda_B}{\lambda_B}-1\right)W^B$$
$$> \left(\frac{\gamma+\lambda_B}{\lambda_B}-1\right)(W^A+W^B) = \frac{\gamma}{\lambda_B}S > 0 \quad (3.104)$$

In addition, when A's wealth increases, in other words as $(1-\eta)Z_t\frac{\lambda_B}{\lambda_A} \to \eta$, we have stock price jumps be come more positive.

But when $(1-\eta)Z_t\frac{\lambda_B}{\lambda_A} < \eta$, stock price jumps can be positive or negative in state 0. In this case

$$\Delta S = \left(\frac{\gamma+\lambda_A}{\lambda_A}-1\right)W^A + \left(\frac{\gamma+\lambda_B}{\lambda_A}-1\right)W^B \quad (3.105)$$

In particular, when $\gamma+\lambda_B < \lambda_A$ and $W^B \geq W^A$ then the stock price jump can be negative. In other words, the dividend goes up and the stock price jumps down.

Recalling that $\gamma + \lambda_B - \lambda_A$ is the lower bound on the real interest rate, we see this possibility corresponds to when the real interest rate can be negative. As $\eta \to 1$, however, stock prices converge to A's private valuation so for large values of η, W^B is small and the stock price jumps will be negative.

3.1.6 Equilibrium Portfolio Choice

The next propostion gives the equilibrium portfolio choice for investor class B.

Proposition 3.8. *Investor B's equilibrium portfolio choice is given by*

$$\theta^B(1,N,t)\Delta S(1,N,t) = \left(\frac{\gamma + \lambda_B(N,t)}{\lambda_t^Q} - 1\right) W^B(1,N,t) - \frac{1}{\lambda_t^Q} 1_{\{(1-\eta)Z_t > \eta\}}$$
$$- \frac{(1-\eta)Z_t}{\lambda_A(N,t)\max(\eta,(1-\eta)Z_t\frac{\lambda_B(N,t)}{\lambda_A(N,t)})} \sum_{j=1}^{\infty} P^B\{N_{t^*(2j,Z_t,t)+t} = 2j + N | N_t = N\}$$
$$- \frac{(1-\eta)Z_t}{\lambda_A(N,t)\max(\eta,(1-\eta)Z_t\frac{\lambda_B(N,t)}{\lambda_A(N,t)})} f^B(1,N,t) \quad (3.106)$$

$$\theta^B(0,N,t)\Delta S(0,N,t) = \left(\frac{\gamma + \lambda_B(N,t)}{\lambda_t^Q} - 1\right) W^B(0,N,t)$$
$$- \frac{(1-\eta)Z_t}{\lambda_A(N,t)\max(\eta,(1-\eta)Z_t\frac{\lambda_B(N,t)}{\lambda_A(N,t)})} \sum_{j=0}^{\infty} P^B\{N_{t^*(2j,Z_t,t)+t} = 2j + 1 + N | N_t = N\}$$
$$- \frac{(1-\eta)Z_t}{\lambda_A(N,t)\max(\eta,(1-\eta)Z_t\frac{\lambda_B(N,t)}{\lambda_A(N,t)})} f^B(0,N,t) \quad (3.107)$$

Proposition 3.8 shows that the portfolio choice for investor class B can be thought of three components: a component which is similar to the case where there is no trade, a component due to the trading of consumption through time, and a component due to learning through time. Recall, we can think of investor B's equilibrium consumption as a stream of option cash flows which pay \$1 in state 1 whenever $(1-\eta)Z_t > \eta$. In other words, this is a cash flow stream of digital options with strike price η. The first sum in Equations (3.106) and (3.107) represent the value of a derivative security which pays \$1 whenvever $(1-\eta)Z_t = \eta$ similar to the hedge ratio for a digital option.

The baseline portfolio choice in the Harrison Kreps equilibrium is fairly static: Investor i holds the asset whenever $\lambda_i(N,t)\Delta S_t > \lambda_j(N,t)\Delta S_t$. When investor B has dogmatic beliefs Proposition 3.8 is consistent with this idea investor class B always

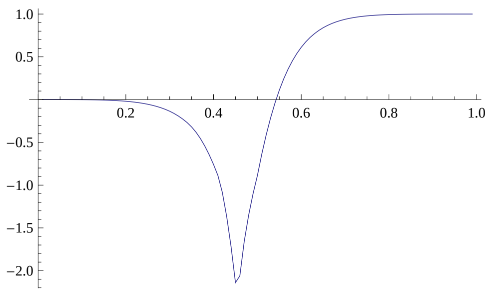

Figure 23: Number of Shares of Stock for Investor Class A in State 1 as a Function of η.
This figure shows how the number of shares Investor class A holds in State 1 as a function of η when investors have dogmatic beliefs. Parameters: $\lambda_A = 0.6$, $\lambda_B = 0.5$, and $\gamma = 0.1$

maintains $\theta_t^B \Delta S_t < 0$. If $\Delta S(1, N, t) < 0$ this means investor class B will always be long the share in state 1. However, investor class A does not always short stock in state 1. When speculators have limited resources however, the wealth distribution introduces changes in the marginal buyer. When investor class B is not very wealthy investor class A will hold shares. Figures 23, 24, 25, and 26 show the equilibrium portfolio choice for Investor class A at time 0 as a function of η when investors have dogmatic beliefs. For large values of η, Investor class A dominates the wealth distribution and in both states Investor class A is buying shares. For smaller values of η Investor class B dominates the wealth distribution and in both states is buying shares. As η goes to $\frac{1}{2}$ the wealth distribution equalizes and we see Investors taking larger speculative positions. We see that investors follow a momentum strategy in one state and a contrarian strategy in the other state. When investors have large disagreements, we see that positions grow unboundedly around the points where $\Delta S = 0$ and their position has reverse sign from the Harrison Kreps strategy in between these points.

Figures 27 and 28 show the initial portfolio choice for investor class A as a function of η when investors learn. The pattern is again similar to the case with no learning because we have fixed $N = t = 0$ so the learning effects are not present.

3.1.7 Term Structure

Recall, equilibrium interest rates are less than or equal to the rate of time preference γ and are lowest when the stock price is just past its peak. The reason for this is that

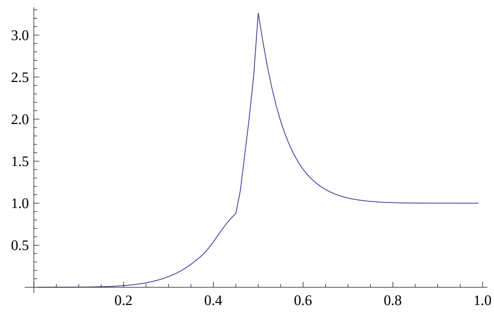

Figure 24: Number of Shares of Stock for Investor Class A in State 0 as a Function of η.
This figure shows how the number of shares Investor class A holds in State 0 as a function of η when investors have dogmatic beliefs. Parameters: $\lambda_A = 0.6$, $\lambda_B = 0.5$, and $\gamma = 0.1$

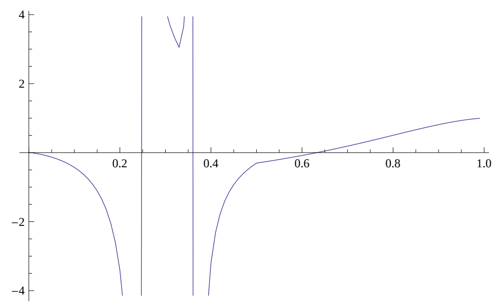

Figure 25: Number of Shares of Stock for Investor Class A in State 1 as a Function of η.
This figure shows how the number of shares Investor class A holds in State 1 as a function of η when investors have dogmatic beliefs for larger disagreement. Parameters: $\lambda_A = 1.0$, $\lambda_B = 0.5$, and $\gamma = 0.1$

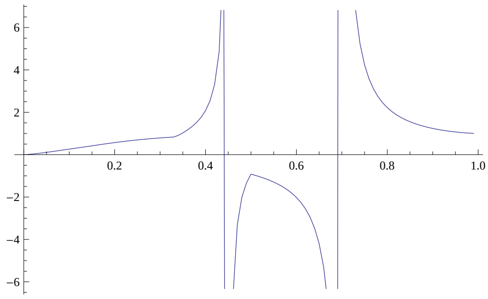

Figure 26: Number of Shares of Stock for Investor Class A in State 0 as a Function of η.
This figure shows how the number of shares Investor class A holds in State 1 as a function of η when investors have dogmatic beliefs for larger disagreement. Parameters: $\lambda_A = 1.0$, $\lambda_B = 0.5$, and $\gamma = 0.1$

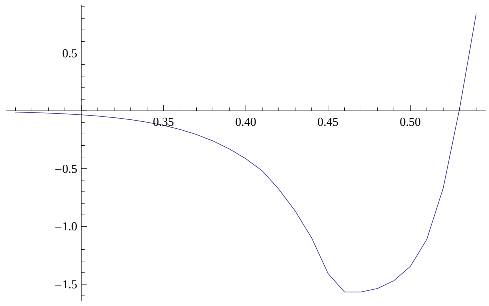

Figure 27: Initial Share Holdings for Agent A in State 1.
This figure shows the number of shares of stock for agent A in state 1 for $N = 0$ versus η when agents learn. Parameters: $z_A = z_B = 1$, $\lambda_A(0, 0) = 0.6$, $\lambda_B(0, 0) = 0, 5$, and $\gamma = 0.1$.

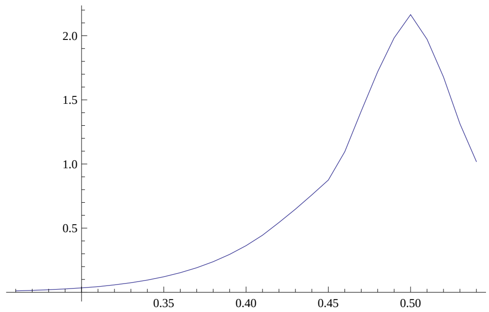

Figure 28: Initial Share Holdings for Agent A in State 0.
This figure shows the number of shares of stock for agent A in state 0 for $N = 0$ versus η when investors learn. Parameters: $z_A = z_B = 1$, $\lambda_A(0, 0) = 0.6$, $\lambda_B(0, 0) = 0, 5$, and $\gamma = 0.1$.

investors account for future speculative investments and anticipate that equilibrium prices will offer even greater speculative prices in the bad states of the world.

Proposition 3.9. *Assume $\lambda_A(N,t) > \lambda_B(N,t)$. A Zero coupon bond which pays one unit and matures at time T has price at time t given by*

$$e^{-\gamma(T-t)}\left(\frac{\eta}{\max[\eta,(1-\eta)Z_t]}\sum_{j=n^*}^{\infty}P^A\{N_T=j|N_t\}+\frac{(1-\eta)Z_t}{\max[\eta,(1-\eta)Z_t]}\sum_{j=0}^{n^*-1}P^B\{N_T=j|N_t\}\right) \tag{3.108}$$

where n^* is given by the smallest nonnegative integer, n, for which

$$\eta P^A\{N_T=n|N_t\} > (1-\eta)Z_t P^B\{N_T=n|N_t\} \tag{3.109}$$

Figure 29 shows the term structure for two different values of η. These correspond to the term structure just after peak stock price and and for the peak stock price. What is interesting is that the term structure is not smooth and does not have to be monotone. Figure 30 compares the term structure of interest rates when the stock price is highest. In both cases we see a steeply upward sloping term structure. The standard model of term structure usually associates a steeply upward sloping term structure with economic expansion. Here we see that a steeply upward sloping term structure can also be associated with wealthy speculators who perceive profitable speculative profits in the future. The term structure is steeper and smootherfor the learning case as opposed to the case where there is no learning. This is because our speculators know that each other will weight the data heavily when revising their estimates and future speculative trade will be less profitable. It is worth noting that these properties will be present in any model in which agents disagree about rare events.

4 Survival

4.1 Survival: Learning

We now examine the survival of agents when they learn from the data. In equilibrium, A consumes when

$$\eta\left(\frac{\Gamma(N_t+z_A)}{\Gamma(z_A)}\frac{k_A^{z_A}}{(t+k_A)^{N_t+z_A}}\right) > (1-\eta)\left(\frac{\Gamma(N_t+z_B)}{\Gamma(z_B)}\frac{k_B^{z_B}}{(t+k_B)^{N_t+z_B}}\right) \tag{4.110}$$

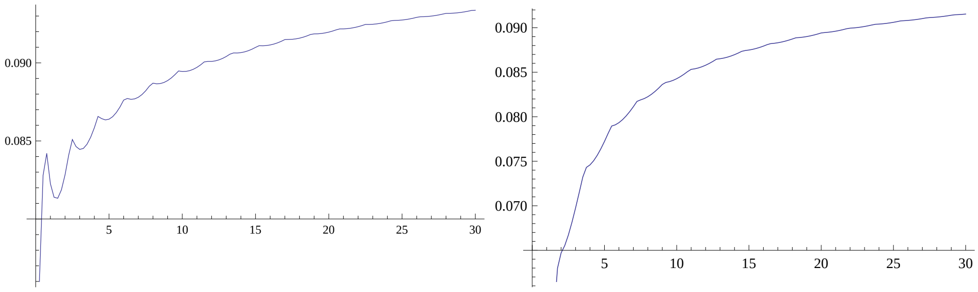

Figure 29: Term Structure.

This figure shows the term structure for parameters $\lambda_A = 0.6$, $\lambda_B = 0.5$, $\gamma = 0.1$. From left to right: $\eta = 0.47$ and $\eta = 0.5$.

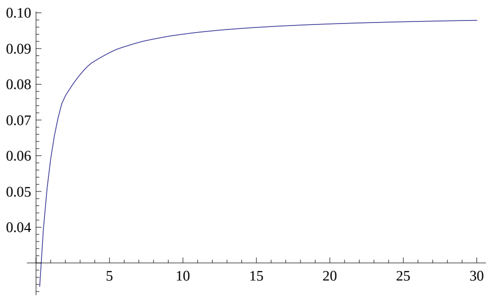

Figure 30: Learning Term Structure.

This figure shows the term structure for $\eta = 0.5$ and $N = 0$. Parameters: $z_A = z_B = 1$, $\lambda_A(0,0) = 0.6$, $\lambda_B(0,0) = 0,5$, and $\gamma = 0.1$.

Rearranging

$$
\begin{aligned}
\frac{\eta}{1-\eta}\frac{\Gamma(z_B)}{\Gamma(z_A)}\frac{k_A^{z_A}}{k_B^{z_B}} &> \frac{\Gamma(N_t+z_B)}{\Gamma(N_t+z_A)}\frac{(t+k_A)^{N_t+z_A}}{(t+k_B)^{N_t+z_B}} \\
&= N_t^{z_A-z_B}\frac{\Gamma(N_t+z_B)}{\Gamma(N_t+z_A)}N_t^{z_B-z_A}t^{z_A-z_B}\left(\frac{(1+\frac{k_A}{t})}{(1+\frac{k_B}{t})}\right)^{N_t}\left(\frac{(1+\frac{k_A}{t})^{z_A}}{(1+\frac{k_B}{t})^{z_B}}\right) \\
&= N_t^{z_A-z_B}\frac{\Gamma(N_t+z_B)}{\Gamma(N_t+z_A)}\left(\frac{N_t}{t}\right)^{z_B-z_A}\exp\left(\frac{N_t}{t}\left(t\log\left(1+\frac{k_A}{t}\right)-t\log\left(1+\frac{k_B}{t}\right)\right)\right)\frac{(1+\frac{k_A}{t})^{z_A}}{(1+\frac{k_B}{t})^{z_B}}
\end{aligned}
\tag{4.111}
$$

Using Abramowitz and Stegun(1970) Equation 6.1.46

$$\lim_{n\to\infty} n^{a-b}\frac{\Gamma(n+b)}{\Gamma(n+a)} = 1 \tag{4.112}$$

and the fact

$$\lim_{t\to\infty} t\log(1+\frac{k_i}{t}) = k_i \tag{4.113}$$

and $\frac{N_t}{t}\to\lambda$, we see that A consumes in the limit when

$$\frac{\eta}{1-\eta}\frac{\Gamma(z_B)}{\Gamma(z_A)}\frac{k_A^{z_A}}{k_B^{z_B}} > \lambda^{z_B-z_A}\exp[\lambda(k_A-k_B)] \tag{4.114}$$

and B consumes in the limit when

$$\frac{\eta}{1-\eta}\frac{\Gamma(z_B)}{\Gamma(z_A)}\frac{k_A^{z_A}}{k_B^{z_B}} < \lambda^{z_B-z_A}\exp[\lambda(k_A-k_B)] \tag{4.115}$$

This can be rearranged to get

Proposition 4.1. *Suppose the true intensity of the Poisson process is λ Then A survives in the limit if*

$$\eta\frac{k_A^{z_A}}{\Gamma(z_A)}e^{-\lambda k_A}\lambda^{z_A-1} > (1-\eta)\frac{k_B^{z_B}}{\Gamma(z_B)}e^{-\lambda k_B}\lambda^{z_B-1} \tag{4.116}$$

and investor class B survives if

$$\eta\frac{k_A^{z_A}}{\Gamma(z_A)}e^{-\lambda k_A}\lambda^{z_A-1} < (1-\eta)\frac{k_B^{z_B}}{\Gamma(z_B)}e^{-\lambda k_B}\lambda^{z_B-1} \tag{4.117}$$

In contrast to the results in Blume and Easley(2006) where agents satisfy Inada conditions and all Bayesian learners survive if the support of their prior contains the truth, survival of risk neutral Bayesian learners depends on the initial wealth distribution as well as their prior density evaluated at the truth. For example if $\eta = 1-\eta$ so prices are maximized at time 0, then the agent whose density evaluated at the truth is highest survives. If, additionally, $z_A = z_B$, we see the investor with the highest k_i survives. That is the investor with the lowest prior expectation and variance of λ will survive.

4.2 Survival: No Learning

Now suppose the true probability measure is P and under P, the intensity of the Poisson process N is given by λ and $\lambda_A > \lambda_B$(the opposite case is simply a relabeling exercise). From the strong law of large numbers we know $\lim_{t\to\infty} \frac{N_t}{t} = \lambda$, P almost surely. Therefore since $\lim_{t\to\infty} \frac{n^*(t)}{t} = \frac{\lambda_A - \lambda_B}{\log(\lambda_A) - \log(\lambda_B)}$, we can deduce A survives if

$$\frac{\lambda_A - \lambda_B}{\log(\lambda_A) - \log(\lambda_B)} < \lambda \tag{4.118}$$

and B surivives if

$$\frac{\lambda_A - \lambda_B}{\log(\lambda_A) - \log(\lambda_B)} > \lambda \tag{4.119}$$

For $0 < y < x$ we have the inequality

$$\sqrt{xy} < \frac{x - y}{\log(x) - \log(y)} < \frac{x + y}{2} \tag{4.120}$$

so a sufficient condition for A to survive is

$$\frac{\lambda_A + \lambda_B}{2} < \lambda \tag{4.121}$$

and a sufficient condition for B to survive is

$$\sqrt{\lambda_A \lambda_B} > \lambda \tag{4.122}$$

which leads to

Proposition 4.2. *If $\lambda_A \neq \lambda_B$ and $\lambda_A = \lambda$ then A survives and B does not. If $\lambda_A \neq \lambda_B$ and $\lambda_B = \lambda$ then B survives and A does not. In other words the investor class with the correct beliefs always survives and in this case the investor class with the wrong beliefs never survives in a Pareto Optimal Allocation with strictly positive planner weights. However, from the individual's perspective, they always belive they will survive and the other investor class will not, that is*

$$\lim_{t\to\infty} P^B\{N_t \leq n^*(t)\} = 1 \tag{4.123}$$

$$\lim_{t\to\infty} P^A\{N_t > n^*(t)\} = 1 \tag{4.124}$$

The last statement follows from $\lim_{t\to\infty} \frac{N_t}{t} = \lambda_A$, P^A almost surely and $\lim_{t\to\infty} \frac{N_t}{t} = \lambda_B$, P^B almost surely.

Recall in the case with learning, survival depended on the wealth distribution as well as investors prior densities evaluated at the truth. In the case with no learning, as long as agents have positive wealth, regardless of the wealth distribution, the agent whose beliefs are in a sense closest to the truth survive. This is because their prior distribution is degenerate. Asymptotically, agents concentrate their wealth into these degenerate states.

5 Conclusion

In this paper we examined several variants of speculative models with and without learning in the framework of a simple continuous time markov chain. Models with unlimited capital along the lines of Harrison and Kreps(1978) do not capture the effects of limited capital on asset prices. When we examine a version of this model in which interest rates are endogenous, many interesting findings arise. First, asset prices are higher than individuals private valuations when the wealth distribution is reasonably similar across individuals. The speculative premium is driven by a disagreement effect and also an interest rate effect. The interest rate effect is due to precautionary speculation – the idea that adverse shifts in states will produce even better speculative prices. Many of our results can be generalized to more complex settings. The basic findings would be robust to many of these generalizations however.

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Appendix

Proof of Proposition 2.1. The bounds follow from the inequality

$$\frac{x}{x+1-a} < e^x x^{1-a}\Gamma(a,x) < \frac{x+1}{x+2-a} \tag{5.125}$$

which is valid for $a < 1$ and $x > 0$. See the NIST Digital Library of Mathematical Functions. $\square$

Proof of Proposition 2.2. Supppose there exist bounded stopping times σ and τ with $\sigma < \tau$ and

$$E^i\left[e^{-\gamma\tau}S_\tau + \int_0^\tau e^{-\gamma s}\delta_s ds|\mathcal{F}_\sigma\right] > e^{-\gamma\sigma}S_\sigma + \int_0^\sigma e^{-\gamma s}\delta_s ds \tag{5.126}$$

or equivalently

$$E^i\left[e^{-\gamma(\tau-\sigma)}S_\tau + \int_\sigma^\tau e^{-\gamma(s-\sigma)}\delta_s ds|\mathcal{F}_\sigma\right] > S_\sigma \tag{5.127}$$

which implies

$$E^i\left[e^{-\gamma(\tau-\sigma)}(S_\tau - e^{\gamma(\tau-\sigma)}S_\sigma) + \int_\sigma^\tau e^{-\gamma(s-\sigma)}\delta_s ds|\mathcal{F}_\sigma\right] > 0 \tag{5.128}$$

The trade borrow S_σ and buy one share of stock at time σ; liquidate the position at time τ, deposit the proceeds in the riskless asset and consume $\gamma(S_\tau - e^{\gamma(\tau-\sigma)}S_\sigma)$ forever is feasible as an incremental trade from any candidate optimum. Notice from time τ on wealth is constant and equal to $S_\tau - e^{\gamma(\tau-\sigma)}S_\sigma$ which satisfies the transversality constraint. Since this trade can be undertaken at any fixed scale there cannot be an optimal solution. Therefore we must have for all bounded stopping times σ and τ with $\sigma < \tau$

$$E^i\left[e^{-\gamma\tau}S_\tau + \int_0^\tau e^{-\gamma s}\delta_s ds|\mathcal{F}_\sigma\right] \le e^{-\gamma\sigma}S_\sigma + \int_0^\sigma e^{-\gamma s}\delta_s ds \tag{5.129}$$

so $e^{-\gamma t}S_t + \int_0^t e^{-\gamma s}\delta_s ds$ is a P^i supermartingale. Therefore

$$S(\iota,0,0) \ge E^i\left[\int_0^s e^{-\gamma t}\delta_t dt\right] + E^i\left[e^{-\gamma s}S_s\right] \tag{5.130}$$

Taking limits, we have

$$S(\iota,0,0) \ge E^i\left[\int_0^\infty e^{-\gamma t}\delta_t dt\right] = S^i(\iota,0,0) \tag{5.131}$$

$\square$

Proof of Proposition 2.3. Obviously the stock market clears given the trading strategies above so the only thing to show is that this is optimal for each investor. For an arbitrary feasible choice of c, θ, α we have

$$\begin{aligned} & e^{-\gamma t} W_t + \int_0^t e^{-\gamma s} c_s ds \\ & = W_0^i + \int_0^t e^{-\gamma s} (\lambda_i(N_s, s) - \lambda^Q(N_s, s)) \theta_s \Delta S_s ds + \int_0^t e^{-\gamma s} \theta_s \Delta S_s (dN_s - \lambda_i(N_s, s) ds) \end{aligned} \tag{5.132}$$

Taking expectations, then limits as $t \to \infty$, an application of the monotone convergence theorem, using the transversality constraint, and using the fact that

$$E^i \left[\int_0^t e^{-\gamma s} (\lambda_i(N_s, s) - \lambda^Q(N_s, s)) \theta_s \Delta S_s ds \right] \leq 0 \tag{5.133}$$

with equality for the strategy given in the proposition, we obtain

$$W_0^i \geq \sup_{\{c,\theta,\alpha\}} E^i \left[\int_0^\infty e^{-\gamma s} c_s ds \right] \tag{5.134}$$

with equality for the policy described in the Proposition. The transversality constraint holds for the consumption plan in Equation (2.34) because

$$E^i \left[e^{-\gamma t} W_t^i \right] + E^i \left[\int_0^t e^{-\gamma s} c_s^i ds \right] = W_0^i \tag{5.135}$$

Taking limits and applying the monotone convergence theorem and using Equation (2.34) then gives $\lim_{t \to \infty} E^i \left[e^{-\gamma t} W_t^i \right] = 0$. Therefore, the claimed strategy is indeed optimal.

Moreover, the value function at the optimum is strictly higher than that corresponding to never trading and simply consuming the dividends, that is

$$\theta^i S(\iota, 0, 0) > \theta^i S^i(\iota, 0, 0) \tag{5.136}$$

which implies the speculative premium is strictly positive for each i. □

Proof of Proposition 2.4. Let $\theta_t = \frac{\Delta S_t^Q}{\Delta S_t}$, $c_t = \delta_t$, and $W_0 = S^Q(\iota, 0, 0)$. Then the wealth process from these choices satisfies

$$\begin{aligned} dW_t &= (\gamma W_t - \theta_t \lambda_t^Q \Delta S_t - c_t) dt + \theta_t \Delta S_t dN_t \\ &= (\gamma W_t - \lambda_t^Q \Delta S_t^Q - \delta_t) dt + \Delta S_t^Q dN_t \end{aligned} \tag{5.137}$$

Given $W_0 = S^Q(\iota, 0, 0)$ we then conclude $W_t = S^Q(\iota, N_t, t)$ almost surely. Therefore this process provides the same dividends for a lower initial investment. $\square$

Proof of Proposition 2.5. One can easily verify these are solutions to Equations (2.32) and (2.33). Since they are bounded, they must be the minimal consistent prices. $\square$

Proof of Propostion 2.6. Let $\tilde{\lambda}^Q(N,t)$ be the intensity of the counting process in the measure which prices $\tilde{S}$. We know that

$$\begin{aligned} e^{-\gamma u}\tilde{S}^Q(\iota_u, N_u, u) &+ \int_t^u e^{-\gamma s}\delta_s ds \\ &= e^{-\gamma t}\tilde{S}^Q(\iota_t, N_t, t) + \int_t^u e^{-\gamma s}\Delta\tilde{S}^Q(\iota_s, N_s, s)(dN_s - \tilde{\lambda}^Q(N_s, s)ds) \\ &= e^{-\gamma t}\tilde{S}^Q(\iota_t, N_t, t) + \int_t^u e^{-\gamma s}(\lambda^Q(N_s, s) - \tilde{\lambda}^Q(N_s, s))\Delta\tilde{S}^Q(\iota_s, N_s, s)ds \\ &\qquad + \int_t^u e^{-\gamma s}\Delta\tilde{S}^Q(\iota_s, N_s, s)(dN_s - \lambda^Q(N_s, s)ds) \end{aligned} \quad (5.138)$$

Our assumptions imply $\lambda^Q(N,s) \leq \tilde{\lambda}^Q(N,s)$ when $\Delta\tilde{S}^Q > 0$ and $\lambda^Q(N,s) \geq \tilde{\lambda}^Q(N,s)$ when $\Delta\tilde{S}^Q < 0$ when $s \geq t^*$ and $N \geq N^*$. Therefore the process

$$e^{-\gamma u}\tilde{S}^Q(\iota_u, N_u, u) + \int_t^u e^{-\gamma s}\delta_s ds \quad (5.139)$$

is a supermartingale in the measure which prices S^Q for $t \geq t^*$ and $N_t \geq N^*$. This, and the boundedness of the minimal consistent prices then imply

$$\tilde{S}^Q(\iota_t, N_t, t) \geq E^Q\left[\int_t^\infty e^{-\gamma(s-t)}\delta_s ds|\mathcal{F}_t\right] = S^Q(\iota_t, N_t, t) \quad (5.140)$$

$\square$

Proof of Proposition 2.7. We first show the prices

$$\tilde{S}^Q(1, N+M, s+t) = \frac{1}{\gamma(B(t)+1)}e^{X(t,s,N)}\left(X(t,s,N)\right)^{z_B+N+M}\Gamma\left(1 - z_B - N - M, X(t,s,N)\right) + \frac{B(t)}{\gamma(B(t)+1)} \quad (5.141)$$

and

$$\tilde{S}^Q(0, N+M, s+t) = \frac{1}{\gamma(B(t)+1)}e^{X(t,s,N)}\left(X(t,s,N)\right)^{z_B+N+M}\left(\frac{A(t,N)-\gamma B(t)}{A(t,N)+\gamma}\right)\Gamma\left(1 - z_B - N - M, X(t,s,N)\right) + \frac{B(t)}{\gamma(B(t)+1)} \quad (5.142)$$

and

$$X(t,s,N) = \frac{(A(t,N)+\gamma)(k_B+t+s)}{B(t)+1} \quad (5.143)$$

are the minimal consistent prices for a Harrison Kreps equilibrium which starts at time t when $N_t = N$ and investor class A estimates the intensity to be $\tilde{\lambda}_A(N+M, s+t) = A(t,N) + B(t)\lambda_B(N+M, s+t)$. Then the bounds follow from Proposition 2.6 since $\tilde{\lambda}_A(N+M, s+t) \geq \lambda_A(N+M, s+t)$ for $s \geq 0$ and $M \geq 0$.

To do this, first observe that under the assumption $z_A \leq z_B + \gamma(k_B + t)$ then $A(t,N) \leq \gamma B(t)$ and

$$\tilde{S}^Q(0, N+1, t+s) < \frac{B(t)}{\gamma(1+B(t))} < \tilde{S}^Q(1, N, t+s) \tag{5.144}$$

and

$$\tilde{S}^Q(0, N, t+s) < \frac{B(t)}{\gamma(1+B(t))} < \tilde{S}^Q(1, N+1, t+s) \tag{5.145}$$

so $\Delta\tilde{S}(0, N, t+s) > 0$ and $\Delta\tilde{S}(1, N, t+s) < 0$. We next take the derivative of these prices with respect to s. After using the recursion $\Gamma(-z_B - N - M, x) = -\frac{1}{z_B+N+M}\Gamma(1 - z_B - N - M, x) + \frac{1}{z_B+N+M}e^{-x}x^{-z_B-N-M}$ and tedious algebra we find

$$\frac{\partial \tilde{S}^Q(1, N+M, t+s)}{\partial s} = \gamma\tilde{S}^Q(1, N+M, t+s) - \lambda_B(N, t+s)(\tilde{S}^Q(0, N+M+1, t+s) - \tilde{S}^Q(1, N+M, t+s)) \tag{5.146}$$

$$\frac{\partial \tilde{S}^Q(0, N+M, t+s)}{\partial s} = \gamma\tilde{S}^Q(0, N+M, t+s) - \tilde{\lambda}_A(N, t+s)(\tilde{S}^Q(1, N+M+1, t+s) - \tilde{S}^Q(0, N+M, t+s)) \tag{5.147}$$

so these satisfy Equations (2.32) and (2.33) and the fact these prices are bounded implies the prices are the Harrison Kreps minimal consistent prices from Propostition 2.3.

The remaining bounds follow as in Proposition 2.1. □

Proof of Proposition 2.8. Follows from Equation (2.56), the fact $\tilde{S}^Q(\iota, N, t) \geq S^B(\iota, N, t)$ and $\frac{N_t}{t} \to \lambda$ P^A or P^B almost surely. □

Proof of Proposition 2.9. We have $\int_0^\infty e^{-\gamma t}(c_t^A + c_t^B)dt = S(\iota, 0, 0)$ and by definition, $S^i(\iota, 0, 0) = E^i\left[\int_0^\infty e^{-\gamma t}\delta_t dt\right]$ which gives the first equality in Equation (2.66). Taking the expectation under P^i and the limit as $t \to \infty$ in Equation (2.64) gives the second equality in Equation (2.66). The third equality in Equation (2.66) follows from

$$\begin{aligned} & e^{-\gamma t}S_t + \int_0^t e^{-\gamma s}\delta_s ds \\ & \quad = S_0 + \int_0^t e^{-\gamma s}(\lambda_i(N_s, s) - \lambda^Q(N_s, s))\Delta S_s ds + \int_0^t e^{-\gamma s}\Delta S_s\left(dN_s - \lambda_i(N_s, s)ds\right) \end{aligned} \tag{5.148}$$

and the fact that

$$\int_0^t e^{-\gamma s}(\lambda_i(N_s,s)-\lambda^Q(N_s,s))\Delta S_s ds = \int_0^t e^{-\gamma s}(\lambda_i(N_s,s)-\lambda_j(N_s,s))\theta_s^j \Delta S_s \quad (5.149)$$

Taking expectations under P^i in Equation (5.148), the limit as $t\to\infty$ and using Equation (5.149) then gives the third equality. □

Proof of Proposition 2.10. We have $\int_0^\infty e^{-\gamma t}(c_t^A + c_t^B)dt = S(\iota,0,0)$ and by definition, $S^Q(\iota,0,0) = E^Q\left[\int_0^\infty e^{-\gamma t}\delta_t dt\right]$ which gives the first equality in Equation (2.70). Taking the expectation under Q and the limit as $t\to\infty$ in Equation (2.64) gives the second equality in Equation (2.70). □

Proof of Proposition 3.1. The form of the value function given in Equation (3.72) follows from standard arguments. Given this form, assume there exists an optimal solution. Then $e^{-\gamma t}W_t h^i(N_t,t) + \int_0^t e^{-\gamma s}c_s ds$ is a martingale for the optimal policy and a supermartingale for any other policy. Applying integration by parts to this gives

$$\begin{aligned}
& de^{-\gamma t}W_t h^i(N_t,t) + e^{-\gamma t}c_t dt \\
&= e^{-\gamma t}h^i(N_t,t)dW_t + e^{-\gamma t}W_t dh^i(N_t,t) + e^{-\gamma t}\Delta W_t \Delta h^i(N_t,t)dN_t - \gamma e^{-\gamma t}W_t h^i(N_t,t) + e^{-\gamma t}c_t dt \\
&= e^{-\gamma t}\left(\frac{\partial h}{\partial t}W_t + h^i(N,t)\left\{(r_t-\gamma)W_t + \left(\lambda_i(N,t)-\lambda^Q(N,t)\right)\theta_t\Delta S_t\right\} + (1-h^i(N,t))c_t\right)dt \\
&\quad + e^{-\gamma t}\left(\lambda_i(N,t)h^i(N+1,t)W_t + \lambda_i(N,t)\left(h^i(N+1,t)-h^i(N,t)\right)\theta_t\Delta S_t\right)dt \\
&\quad + e^{-\gamma t}\left(h^i(N,t)\theta_t\Delta S_t + W_t\Delta h_t^i + \Delta h_t^i \Delta W_t\right)(dN_t - \lambda_i(N,t)dt) \quad (5.150)
\end{aligned}$$

which implies

$$\begin{aligned}
& W_t\left(\frac{\partial h}{\partial t} + h^i(N,t)(r_t-\gamma-\lambda_i(N,t)) + \lambda_i(N,t)h^i(N+1,t)\right) \\
&\quad + \left\{\lambda_i(N,t)h^i(N+1,t) - \lambda^Q(N,t)h^i(N,t)\right\}\theta_t\Delta S_t + (1-h^i(N,t))c_t \le 0 \quad (5.151)
\end{aligned}$$

$$\begin{aligned}
& W_t\left(\frac{\partial h}{\partial t} + h^i(N,t)(r_t-\gamma-\lambda_i(N,t)+\lambda^Q(N,t))\right) \\
&\quad + \left\{\lambda_i(N,t)h^i(N+1,t) - \lambda^Q(N,t)h^i(N,t)\right\}(\theta_t\Delta S_t + W_t) + (1-h^i(N,t))c_t \le 0 \\
&\quad (5.152)
\end{aligned}$$

with equality when θ_t and c_t are optimal. The proposition then follows from the fact that $c_t \ge 0$ and $W_t + \theta_t\Delta S_t \ge 0$. □

Proof of Corollary 3.1. If there is an optimal solution where the wealth constraint does not bind, then

$$\lambda_i(N,t)h^i(N+1,t) = \lambda^Q(N,t)h^i(N,t) \quad (5.153)$$

The corollary then follows by integration by parts. $\square$

Proof of Proposition 3.2. We have $W_t^j \leq S_t^j$ and thus $W_{t-}^j + \theta_t^j \Delta S_t \leq S_{t-}^j + \Delta S_t^j$. From market clearing, $W_{t-}^j = S_{t-} - W_{t-}^i$ so $S_{t-} - W_{t-}^i + \theta_t^j \Delta S_t \leq S_{t-}^j + \Delta S_t^j$. Again from market clearing, $\theta_t^i = 1 - \theta_t^j$, so $S_{t-} - S_{t-}^j + \Delta S_t - \Delta S_t^A \leq W_{t-}^i + \theta_t^i \Delta S_t$. This gives Equation (3.79).

If $W_{t-}^j > S_t^j$ then this and Equation (3.79) give $W_{t-}^i + \theta_t^i \Delta S_t \geq S_t - S_t^j > S_t - W_{t-}^j$ and hence $S_{t-} + \theta_t^i \Delta S_t > S_t$. Therefore if $\Delta S_t < 0$, $\theta_t^i < 1$ so $\theta_t^j > 0$. $\square$

Proof of Proposition 3.3. We will prove this for the case where investors learn from the data. The case where investors have dogmatic beliefs is identical and simpler.

Suppose, to the contrary, that the wealth constraint binds for investor i, that is there exists an N and an open interval $\mathcal{U}$ such that in equilibrium, $W_u = 0$ for all $u \in \mathcal{U}$. In this case, for any $u \in \mathcal{U}$, investor j will set future prices so $r_t \equiv \gamma$ and $\lambda^Q(N,t) = \lambda_j(N,t)$ for all $t \geq u$. To alleviate notation, define $\lambda_u^i(N,t) \equiv \lambda_i(N, t+u)$.

Consider the feasible trading strategy which starts with $W_u > 0$, with no consumption withdrawals $c_t = 0$ and fixes a $\pi_t = \frac{\theta_t S_{t-}}{W_{t-}}$, with $1 + \pi_t \frac{\Delta S^j(\iota,N,t)}{S^j(\iota,N,t))} > 0$ and let $\theta_t = \frac{\pi_t W_t}{S^j(\iota,N,t)}$. Define τ as the time the state shifts from ι to $1-\iota$. Then at time τ we have

$$e^{-\gamma\tau} W_\tau = W_u \exp\left[-\int_0^\tau \lambda_u^j(N,s) \frac{\pi_s \Delta S^j(\iota,N,s+u)}{S^j(\iota,N,s+u)} ds\right] \left(1 + \pi_\tau \frac{\Delta S^j(\iota,N,\tau+u)}{S^j(\iota,N,\tau+u)}\right) \tag{5.154}$$

Therefore

$$\begin{aligned}
&E^i\left[e^{-\gamma\tau} W_\tau | N_u = N\right] \\
&= W_u \int_0^\infty \lambda_u^i(N,t) \left(1 + \pi_t \frac{\Delta S^j(\iota,N,t+u)}{S^j(\iota,N,t+u)}\right) \exp\left[\int_0^t \left(-\lambda_u^j(\iota,N,s)\pi_s \frac{\Delta S^j(\iota,N,s+u)}{S^j(\iota,N,s+u)} - \lambda_u^i(N,s)\right) ds\right] dt \\
&\qquad = W_u \int_0^\infty \lambda_u^i(N,t) \left(1 + \pi_t \frac{\Delta S^j(\iota,N,t+u)}{S^j(\iota,N,t+u)}\right) \\
&\times \exp\left[\int_0^t \left(\left(\lambda_u^i(N,s) - \lambda_u^j(N,s)\right) \pi_s \frac{\Delta S^j(\iota,N,s+u)}{S^j(\iota,N,s+u)} - \lambda_u^i(N,s)\left(1 + \pi_s \frac{\Delta S^j(\iota,N,s+u)}{S^j(\iota,N,s+u)}\right)\right) ds\right] dt
\end{aligned} \tag{5.155}$$

In particular, for $i = A$ set $\pi_t = \frac{\ln(K)}{\lambda_u^A(N,t) - \lambda_u^B(N,t)} \frac{S^B(\iota,N,t+u)}{\Delta S^B(\iota,N,t+u)}$. Then $1 + \pi_t \frac{\Delta S^B(\iota,N,t+u)}{S^B(\iota,N,t+u)} > 0$ for any $K > 1$ and

$$\exp\left[\int_0^t \left(\left(\lambda_u^A(N,s) - \lambda_u^B(N,s)\right) \pi_s \frac{\Delta S^j(\iota,N,s+u)}{S^j(\iota,N,s+u)}\right) ds\right] = e^{t\ln(K)} = K^t \tag{5.156}$$

so

$$
\begin{aligned}
&E^A\left[e^{-\gamma\tau}W_\tau|N_u=N\right]\\
&=W_u\int_0^\infty K^t\lambda_u^A(N,t)\left(1+\pi_t\frac{\Delta S^j(\iota,N,t+u)}{S^j(\iota,N,t+u)}\right)\exp\left[-\int_0^t\lambda_u^A(\iota,N,s)\left(1+\pi_s\frac{\Delta S^j(\iota,N,s+u)}{S^j(\iota,N,s+u)}\right)ds\right]dt
\end{aligned}
\tag{5.157}
$$

Since any $K>1$ is possible, this implies the value function for investor A is infinite whenever prices are set by investor B. This is inconsistent with equilibrium; investor A can improve on any candidate optimum by avoiding default, setting consumption equal to 0 until time τ, consume $\frac{W_\tau}{\gamma}$ and enjoy arbitrarlity high expected utility. We conclude investor class A will never default in equilibrium.

When $i=B$, set $\pi_t=-\frac{\lambda_u^B(N,t)}{\lambda_u^A(N,t)}\frac{S^A(\iota,N,s+u)}{\Delta S^A(\iota,N,s+u)}$. Then $1+\pi_t\frac{\Delta S^A(\iota,N,t+u)}{S^A(\iota,N,t+u)}>0$ and

$$
\begin{aligned}
E^B\left[e^{-\gamma\tau}W_\tau|N_u=N\right]&=W_u\int_0^\infty\lambda_u^B(N,t)\left(1-\frac{\lambda_u^B(N,t)}{\lambda_u^A(N,t)}\right)dt\\
&=W_u\int_0^\infty\frac{z_B+N}{t+u+k_B}\left(\frac{(t+u+k_B)(z_A+N)-(t+u+k_A)(z_B+N)}{(t+u+k_B)(z_A+N)}\right)dt\\
&\geq W_u\int_0^\infty\frac{z_B+N}{t+u+k_B}\left(\frac{(t+u+k_B)(z_A+N)-(t+u+k_B)(z_B+N)}{(t+u+k_B)(z_A+N)}\right)dt\\
&=W_u\frac{z_B+N}{z_A+N}(z_A-z_B)\int_0^\infty\frac{1}{t+u+k_B}dt=\infty
\end{aligned}
\tag{5.158}
$$

Since $k_B\geq k_A$ and $z_B<z_A$. For a similar reason as before, this is inconsistent with equilibrium. □

Proof of Proposition 3.4. From Proposition 3.1 we know market clearing in the consumption market requires $h^i(N,t)=1$ for some i. Since $h^i(N,t)=1$ implies $r_t+\lambda^Q(N,t)=\gamma+\lambda_i(N,t)$ and $\lambda_A(N,t)>\lambda_B(N,t)$ for all N and t we know that (except possibly for a measure 0 set of points), if $r_t+\lambda^Q(N,t)=\gamma+\lambda_i(N,t)$, then $h^j(N,t)>1$ for $j\neq i$. Therefore an equilibrium is characterized by only one investor consumes the entire dividend in each time and state while the other investor saves.

Therefore

$$
J^i(W_t^i,N_t,t)=W_t^ih^i(N_t,t)=E^i\left[\int_t^\infty e^{-\gamma(s-t)}\delta_s 1_{\{h^i(N_s,s)=1\}}ds|\mathcal{F}_t\right]
\tag{5.159}
$$

which gives

$$
J^i(W_t^i,N_t,t)\equiv W^i(\iota,N_t,t)=W_t^i=\frac{1}{h^i(N_t,t)}E^i\left[\int_t^\infty e^{-\gamma(s-t)}\delta_s 1_{\{h^i(N_s,s)=1\}}ds|\mathcal{F}_t\right]
\tag{5.160}
$$

and the equilibrium stock price is given by market clearing.

$$S(\iota, N_t, t) = W_t^A + W_t^B \tag{5.161}$$

So determining equilibrium boils down to determining the functions $h^A(N,t)$ and $h^B(N,t)$.

For some arbitrary $\eta \in (0,1)$, let $h^A(N_t,t) = \frac{\max(\eta,(1-\eta)Z_t)}{\eta}$ and let $h^B(N_t,t) = \frac{\max(\eta,(1-\eta)Z_t)}{(1-\eta)Z_t}$. For each fixed N we have $h^A(N,t) > 1$ implies $h^B(N,t) = 1$ and $h^B(N,t) > 1$ implies $h^A(N,t) = 1$ except for points where $\eta = (1-\eta)Z_t$. When $\lambda^A(N,t) > \lambda^B(N,t)$, $h^A(N,t)$ is strictly increasing in t when $\eta < (1-\eta)Z_t$ while $h^B(N,t)$ is strictly decreasing in t when $\eta > (1-\eta)Z_t$ so these points are lesbesgue measure 0. Given these choices, define $\theta(\eta)$ by

$$\theta(\eta) = \frac{W^A(\iota,0,0)}{S(\iota,0,0)} \tag{5.162}$$

and $\rho_t^A = e^{-\gamma t}\frac{h^A(N_t,t)}{h^A(0,0)}$. Equate $\rho_t^A = e^{-\int_0^t r_s ds}M_t$ for a process M_t which satisfies $dM_t = \Delta M_t(dN_t - \lambda^i(N_t,t)dt)$, $M_0 = 1$. Integration by parts and matching coefficients then identifies the interest rate r_t. The relation $h^i(N,t) = 1 \Rightarrow r_t + \lambda^Q(N,t) = \gamma + \lambda_i(N,t)$ identifies λ^Q. This then defines an equilibrium for the initial endowments $\theta(\eta) = \theta^A$ and $1-\theta(\eta) = \theta^B$. By varying η we can find an equilibrium for any initial endowments. To see this observe that $\lim_{\eta\to 0}\theta(\eta) = 0$, $\lim_{\eta\to 1}\theta(\eta) = 1$, and $\theta(\eta)$ is continuous in η. □

Proof of Proposition 3.5. The stock price is given by

$$S_t = \frac{1}{\rho_t^A}E^A\left[\int_t^\infty \rho_s^A \delta_s ds|\mathcal{F}_t\right] \tag{5.163}$$

The expressions in the proposition come from interchanging the expectation and the integral, and then interchanging the integral and the resulting sum. □

Proof of Proposition 3.6. Investor B's wealth is given by

$$W_t^B = \frac{1}{\rho_t^B}E^B\left[\int_t^\infty \rho_s^B \delta_s 1_{\{h^B(N_s,s)=1\}} ds|\mathcal{F}_t\right] \tag{5.164}$$

where $\rho_t^B = e^{-\gamma t}\frac{h^B(N_t,t)}{h^B(0,0)}$. The expressions in the proposition follow by interchanging the expectation and the integral, and then interchanging the integral and the resulting sum. □

Proof of Corollary 3.2. The expressions follow from taking the appropriate derivatives in Equations (3.87) and (3.88). □

Proof of Proposition 3.7. We have

$$\frac{\partial S(\iota, N, t)}{\partial t} = r_t S_t - \delta_t - \lambda_t^Q \Delta S_t \tag{5.165}$$

or

$$\frac{\Delta S_t}{S_t} = \frac{r_t S_t - \delta_t - \frac{\partial S(\iota, N, t)}{\partial t}}{\lambda_t^Q S_t} \tag{5.166}$$

The result now follows from Corollary 3.2 and tedious algebra. □

Proof of Proposition 3.8. The result follows from differentiating Equations (3.90) and (3.91) with respect to t,

$$\frac{\partial W^B}{\partial t} = r_t W_t^B - c_t^B - \lambda_t^Q \theta_t^B \Delta S_t \tag{5.167}$$

and tedious algebra. □

Proof of Proposition 3.9. The time t price of a zero coupon bond maturing at time T is given by

$$P_t = \frac{1}{\rho_t^A} E^A \left[\rho_t^A | \mathcal{F}_t \right] \tag{5.168}$$

The expression in the proposition comes from evaluating this expression. □