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Full characterization of the nonnegative core of some cooperative games

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We propose a method to design cost allocation contracts that help maintain the stability of strategic alliances among firms by using cooperative game theory. The partners of the alliance increase their efficiency by sharing their assets. We introduce a new sufficient condition for total balancedness of regular games, and a full characterization of their nonnegative core. A regular game is defined by a finite number of resources owned by the players. The initial cost of a player is a function of the vector of quantities of the resources that the player owns. The characteristic function value of a coalition is a symmetric real function of the vectors of its members. Within this class we focus on centralizing aggregation games, meaning that the formation of a coalition is equivalent to aggregating its players into one artificial player whose cost is an intermediate value of the costs of the aggregated players. We prove that under a certain decreasing variation condition, a centralizing aggregation game is totally balanced and its nonnegative core is fully characterized. We present a few nonconcave games in operations management that their nonnegative core is fully characterized, by showing that they satisfy the conditions presented in this article.

KEYWORDS

cooperative games, the core, total balancedness, queueing and scheduling, strategic alliances

1 | INTRODUCTION

This research is related to strategic alliances of independent companies that cooperate by pulling their resources in order to achieve common goals. The partners of the alliance may carry out similar portfolios of activities as is the case with partnerships among airlines that have started to form in 1989 in the collaboration between Northwest and KLM. Today, Star Alliance, the greatest such alliance, consists of 27 companies. Strategic alliances may also be signed among partners that have complementary activities, as for example the ones between Spotify and Uber or between Starbucks and Barnes & Noble. In contrast to mergers or joint ventures where some companies form a larger company by transferring ownership, see for example, AT&T and Time Warner, in strategic alliances the collaborating firms keep their own identity, implying that the decision of a firm to join a strategic alliance is reversible, and therefore, the alliance's stability is of major concern. Thus, a real challenge that is linked to the formation of a successful strategic alliance is the design of a contract that specifies a fair revenue/cost sharing scheme where all its partners are satisfied and no subset of partnering companies feels that it can do better by quitting the alliance and forming a new smaller partnership. In this paper we use the principles of *cooperative game theory with transferrable utilities* to design revenue/cost allocation that supposedly maintain the stability of strategic alliances. Our results fit better strategic alliances among companies that propose similar portfolios of activities and for the sake of reaching a larger market segment, they share their assets, facilities and/or databases.

Any particular application of a strategic alliance among firms has its own characteristics, usually implying the necessity of a tailored made solution technique that derives a revenue/cost allocation that ensures stability. For example, a two-stage game theoretic approach has been proposed for the revenue sharing problem of strategic alliances among airlines, mentioned above (see Hu, Caldentey, & Vulacano, 2013). In this article, we do not refer to a specific type of a business.

Instead, we consider general strategic alliances among service or manufacturing firms that share a similar domain of activities and their common interest is to minimize their operational costs. The firms collaborate by combining their assets and competencies in order to improve their efficiency and their profit. The contribution of each firm to the partnership is a function of the initial amount of the assets that it possesses (we limit ourselves to quantitative assets only). In order to ensure stability of the partnership, the cost allocated to the firms, and to all subsets of firms, that are members of the partnership, should not be higher than their cost prior to forming the collaboration. If such a cost allocation vector exists for a certain partnership, the game is said to be *balanced*. In general, if the game is balanced and the variability among the firms is remarkable, it might happen that the strongest firms, that is, the ones that contribute most of the assets to the partnership will be courted by the weaker ones to the extent that the cost allocation contract that ensures stability, may not only totally waive the costs of the strongest firms but it may also specify payments of the weaker firms to the strongest ones in order to solicit them to join the partnership. Any negative entry in the cost allocation vector represents a payment made to the corresponding strong partner by the weaker partners. In operations management, cost allocation vectors that contain negative entries, even if stability of the partnership is guaranteed, are less attractive in the long run, as they might cause the weak partners to resent the strong ones that are paid, putting the partnership at risk.

This paper considers strategic alliances under which the assets of a number of service or production systems are consolidated into a single "super-server" or "super production facility," in order to increase the partnership's efficiency. Various reasons give rise to the practice of consolidation where the main one is reduction of operational costs. Additional possible benefits of consolidation include the reduction of congestion, reduction of greenhouse gas emissions, ameliorating the service experience of customers in terms of service time and convenience, etc. We prove that under certain conditions, such collaborations give rise to a *totally balanced cooperative game*, that is, the game and all its subgames are balanced. Moreover, we prove that for the type of games considered in this article, there always exists core cost allocation vectors in which there is no need to convince a partner to join the cooperation by paying him, no matter how influential the partner is. In fact, we fully characterize the set of nonnegative cost allocation vectors in the *core*, that is, the nonnegative vectors that fulfill all the conditions of stability, but we do not characterize the part of the core where the vectors contain negative entries.

In the context of service systems, models that minimize the long-run average congestion cost among parallel $M/M/1$ queueing systems by partial consolidation, are analyzed in Anily and Haviv (2017), where systems may cooperate by either reallocating the incoming streams of customers while leaving the servers' capacities intact, or alternatively, by reallocating the total servers' capacities among the servers while keeping the incoming streams of customers intact. In this paper, we focus on full consolidation of parallel $M/M/1$ queueing systems as done in Anily and Haviv (2017), where any coalition of systems combines additively the arrival rates and the service rates of its members in order to form a new "super" $M/M/1$ queueing system whose cost is its long-run congestion. Below and in Subsection 1.1, we refer to this game as the *consolidation game of parallel M/M/1 queueing systems*. A process that has some similarities to the consolidation game of $M/M/1$ systems is the *visa travel service AustraliaETA*, where citizens of 35 countries, including the United States and most of the European countries, request a visitor visa to Australia. Prior to the use of the online service, any citizen of these countries had to come to the closest Australian embassy or consulate, wait in line to fill some information, leave there her passport and wait for the visa by mail. Today, the citizens of all these countries use the same online service to request a visitor visa, which is usually obtained in a few hours.

If the consolidation game of $M/M/1$ systems were concave, then it would be totally balanced and the sufficient condition proposed by Shapley (1971) could be invoked to fully characterize its nonempty core. But, Anily and Haviv (2010) prove that the game is nonconcave, and proposes to link to the game a new concave game called its *auxiliary game* whose core coincides with the nonnegative part of the core of the original game, proving that the original game is totally balanced. The question that has been often raised in the context of the proof in Anily and Haviv (2010) is whether the genuine idea behind the auxiliary game is generalizable beyond the queueing game. In this article we investigate the type of games that can be proved to be totally balanced by using an appropriate auxiliary game. For that sake we shortly review the main concepts of cooperative game theory.

Cooperative games with transferable utilities are coalitional games defined by a pair (N, C) where $N=\{1, \ldots, n\}$ is a set of *players* and the *characteristic function*. C is a set function that returns a real number $C(S)$ for any coalition $\emptyset \subseteq S \subseteq N$, that is, ., $C: 2^N \rightarrow \mathfrak{R}$, where $\mathfrak{R}$ is the set of real numbers, and $C(\emptyset)=0$. We refer here to $C(S)$ as the cost of coalition $S \subseteq N$. The coalition $S=N$ is called the *grand-coalition*. Under any partition of the grand-coalition into disjoint sets $S_1, \ldots, S_m$, the cost of the game is $\sum_{\ell=1}^{m} C(S_\ell)$, meaning that the total cost is additive in the coalitions. A necessary condition for all the players of N to cooperate and form the grand-coalition, is subadditivity of the game: a game (N, C) is *subadditive* if and only if the characteristic function C is *subadditive*, that is, for any two disjoint coalitions S, $T \subset N$, $C(S \cup T) \leq C(S)+C(T)$. Subadditivity implies that $C(N) \leq \sum_{\ell=1}^{m} C(S_\ell)$ for all partitions $\{S_1, \ldots, S_m\}$, $m \geq 1$, of N, meaning that the grand-coalition is an optimal formation of coalitions in terms of total cost.

Once that the grand-coalition is formed, the players bargain for a fair cost allocation scheme of the total cost $C(N)$. Various cost allocation concepts have been proposed, where the common guideline is achieving a reasonable amount of stability or fairness among the players. Let $\hat{x} = (x_1, \ldots, x_n) \in \mathfrak{R}^n$ be a *cost allocation vector* where x_i, $i \in N$, is the cost allocated to player i. The condition $\sum_{i=1}^{n} x_i = C(N)$, called *efficiency* is preliminary for a cost allocation vector. For the sake of this paper we describe the two most renowned concepts of stability or fairness, the core that is attributed to Shapley (1955) (see Zhao, 2017), and the Shapley value. The *core* of the game (N, C) consists of all efficient cost allocation vectors that allocate to the members of any coalition S, $S \subseteq N$, no more than $C(S)$, that is, $\sum_{i \in S} x_i \leq C(S)$. These last requirements are called the *stand-alone* conditions. This set of conditions guarantees stability as no subset of players can claim to reduce its cost by quitting the grand-coalition. The linearity of the conditions in the variables x_i for $i \in N$ implies that the core is either empty, consists of a single vector or is infinitely large. A cooperative game (N, C) whose core is nonempty is said to be *balanced*, and a game whose core and the cores of all its subgames are nonempty, is *totally balanced*. Finding out if a game is balanced, let alone characterizing the whole core of the game, is intricate as the corresponding linear programming formulation is of an exponential size, since a stand-alone constraint is required for any subset of N. Another well-known cost allocation concept is the *Shapley value*. Any cooperative game is associated with a unique value, called its Shapley value. The Shapley value of player $i \in N$ is the average marginal cost of adding the player to the players that precede him where averages are taken with respect to all potential orders of the players. The Shapley value does not necessarily belong to the core of a balanced game, but its definition is plausible and sounds fair. In addition to the efficiency property, the Shapley value satisfies symmetry, linearity, and the null player properties (see Shapley, 1953). In fact, the Shapley value is the only value that satisfies all the above four properties.

Many cooperative games, especially in operations management and logistics, have a further feature that allows a more efficient presentation than listing the 2^n values that the characteristic function assumes. The class of *regular games* is proposed in Anily and Haviv (2014)): a regular game is defined by a finite list of $\kappa \geq 1$ different quantitative resources indexed by $\ell = 1, \ldots, \kappa$ that the players own. Each player is fully characterized by a *vector of properties* of size κ whose ℓth element represents the amount of resource ℓ that the player owns. The cost of a coalition is a symmetric mapping of the vectors of properties of the players in the coalition into the real numbers and it is otherwise independent of the identity of the players. Regular games can be presented in a compact way by stating the form of the mapping as a function of the collection of vectors of properties. As a consequence it allows more flexibility than the classic presentation (N, C) does, since the mapping returns a real value for any collection of κ-vectors even if the vectors are not associated with "real" players of N. Furthermore, manipulating real functions using appropriate algebraic rules in order to prove new properties is simpler and a safer haven than doing the same with set functions. For example, consider the following definition (see Anily & Haviv, 2014) on homogenous of degree p, $p \geq 0$, cooperative games:

> **Definition 1** A regular game is homogeneous of degree p, p ≥ 0, if for any integer m, the cost of cloning m times a collection of vectors of properties, is m^p times the original cost of these vectors of properties.

While cloning players in a regular game is a simple matter, doing so in a nonregular game (N, C) is meaningless as the characteristic function is defined only on coalitions of N. A regular game that is homogenous of degree 0 displays *economies of scale* as the cost of $m \geq 1$ copies of a collection of vectors of properties is the same as the cost of a single copy, thus the cost per player decreases implying that efficiencies are improved by scale. In homogenous of degree 1 games, the cost increases linearly in the number of copies of the players, thus there are not any economies of scale. Homogenous of degree p games for $0 < p < 1$ ($p > 1$) display economies (diseconomies) of scale, as the average cost per player improves (deteriorates). In Anily and Haviv (2014), it is proved that regular subadditive and homogenous of degree 1 games are totally balanced.

A number of classes of games have been proved to be totally balanced. Here, we focus on the prominent class that is central to our paper, and we shortly mention another pivotal class for the sake of comparison.

- **Condition 1.** A game (N, C) is *concave* if its characteristic function is concave, that is, for any two coalitions $S, T \subseteq N$, $C(S \cup T) + C(S \cap T) \leq C(S) + C(T)$. Concave games are sub-additive but not the other way around. It was shown in Shapley (1971) that the core of a concave game possesses $n!$ extreme points, each of which being the marginal contribution vector of the players for one of the $n!$ permutations of the players. In particular, the Shapley value of a concave game is the center of gravity of its core. If the game is monotone and concave, then its core is non-negative, as all its extreme points are non-negative. Otherwise, there exist core cost allocation vectors with negative entries. Concave games are the most structured cooperative games whose core is fully characterized (see Shapley, 1971). As a side remark note that the set of *average concave games*, introduced in Iñara and Usategui (1993), contains the set of concave games. Similarly, they are totally balanced and their core contains their Shapley value.

- **Condition 2.** A *market game*, see for example, Shapley and Shubik (1969) and Chapter 13 in Osborne and Rubinstein (1994), is defined as follows: Suppose there are κ inputs. An *input vector* is a nonnegative vector in $\mathfrak{R}^\kappa$, denoted by $(\mathfrak{R}_0^+)^\kappa$. Each of the n players possesses an initial commitment vector $w_i \in (\mathfrak{R}_0^+)^\kappa$, $1 \le i \le n$, which states a nonnegative quantity for each input. Moreover, each player is associated with a continuous and convex cost function $f_i : (\mathfrak{R}_0^+)^\kappa \to \mathfrak{R}_0^+$, $1 \le i \le n$. A profile $(z_i)_{i\in N}$ of input vectors for which $\sum_{i\in N} z_i = \sum_{i\in N} w_i$ is an *allocation*. The game is such that a coalition S of players seeks an optimal redistribution of its members' commitments among its members in order to get a profile $(z_i)_{i\in S}$ of input vectors that minimizes the total cost across the members of S. Formally, for any $\emptyset \subseteq S \subseteq N$,

$$C(S) = \min\left\{ \sum_{i\in S} f_i(z_i) : z_i \in (\mathfrak{R}_0^+)^\kappa, \quad i \in S \quad \text{and} \quad \sum_{i\in S} z_i = \sum_{i\in S} w_i \right\} \quad (1)$$

In contrast to concave games whose entire core is well defined, just a single core cost allocation, based on *competitive equilibrium prices*, is known for market games (see Osborne & Rubinstein, 1994, p. 266).

We note that neither concave games nor market games are necessarily regular. A market game (N, C) is regular if and only if the characteristic function value of the grand-coalition $C(N)$, (see Anily & Haviv, 2010), is given as the minimum of the sum of n identical functions, that is, $f_i = f$ for all $i \in N$. In such a case, the cost of a vector of properties is independent of the identity of the player that is in possession of the vector. In other words, regular games satisfy the anonymity property, that is, the cost of any coalition depends only on the respective collection of the vectors of properties of its members, and is, otherwise, independent of their identities.

In this article we concentrate on a subclass of regular games that we call *centralizing aggregation games*, where an *aggregation function* aggregates any number of vectors of properties into a new vector of properties. Centralizing means that the cost of the vector of properties generated by aggregating a certain input of vectors of properties, behaves like a measure of centrality of the costs of the individual vectors of the input. More specifically, the cost of the new vector is in between the cost of the cheapest vector and the cost of the most expensive vector in the input, and it is strictly increasing in the cost of the vectors in the input. A precise definition of a centralizing aggregation game is given in Section 3. Note that the aggregated vector is not necessarily associated with a player of N, but nevertheless, as the game is regular, its cost is well defined.

The main theorem of the paper proves that under a certain decreasing variation condition, a nonnegative centralizing aggregation game is totally balanced and its nonnegative core is fully identifiable. This is done by defining an auxiliary nonnegative monotone game whose core is contained in the core of the original game, and showing that the auxiliary game is concave. The concavity of the auxiliary game and its monotonicity imply that its core is nonempty and nonnegative (see Shapley, 1971). Finally, we show that the core of the auxiliary game coincides with the nonnegative core of the original game.

The outline of the paper is as follows: Section 2 generalizes the concept of auxiliary games proposed in Anily and Haviv (2010) to any game. Section 3 presents some notations and preliminaries, and a rigorous definition of regular games, the class of centralizing aggregation games and some of their properties. Section 4 elaborates on the auxiliary games of centralizing aggregation games. Section 5 presents the main theorem that provides a new sufficient condition for total balancedness of centralizing aggregation games. In fact, it is proved that the nonnegative core of such games is fully characterized. The total balancedness of a few nonconcave games in queueing and scheduling is proved in Section 1 by using the main theorem. Section 2 concludes the paper by presenting an open question on the total balancedness of a class of centralizing aggregation games.

2 | THE AUXILIARY GAME

The concept of the *auxiliary game* of the consolidation game of parallel M/M/1 queueing systems has been defined in Anily and Haviv (2010). Here, we generalize this concept to any cooperative game: let (N, C) be a cooperative game where $N = \{1, \ldots, n\}$ is a set of n players, and the set function $C : 2^N \to \mathfrak{R}$, called the *characteristic function*, returns the cost of any coalition $S \subseteq N$, with $C(\emptyset) = 0$. The *auxiliary game* $(N, \widetilde{C})$ of (N, C), is defined by the characteristic function $\widetilde{C}(S) = \min\{C(T) : S \subseteq T \subseteq N\}$.

Theorem 1 *Suppose that (N, C) is a nonnegative game, and consider the set function $\widetilde{C}(S)$ defined above. Then, the following properties hold:*

1. *For any subset $\emptyset \subseteq S \subseteq N$, $\widetilde{C}(S) \le C(S)$.*
2. *$(N, \widetilde{C})$ is a well-defined game with $\widetilde{C}(\emptyset) = 0$ and $\widetilde{C}(N) = C(N)$.*
3. *$(N, \widetilde{C})$ is a monotone game.*
4. *If the game (N, C) is monotone then the game $(N, \widetilde{C})$ coincides with the game (N, C).*

Proof

1. By definition, $\widetilde{C}(S) \le C(S)$ for any $S \subseteq N$.

2. The only condition that needs to be proved in order for $(N, \widetilde{C})$ to be a well defined game on the set of players N, is that $\widetilde{C}(\emptyset) = 0$. First, note that $\widetilde{C}(\emptyset) \geq 0$ as the game (N, C) is non-negative. Next, we show that $\widetilde{C}(\emptyset) \leq 0$. In view of the first item of the theorem $\widetilde{C}(\emptyset) \leq C(\emptyset) = 0$, concluding the proof that $\widetilde{C}(\emptyset) = 0$. Finally, we prove that $\widetilde{C}(N) = C(N)$, by observing that the grand-coalition, N, contains all the players of N thus there is no way of reducing its cost by adding more players to N.
3. Consider two coalitions $\emptyset \subseteq S \subset T \subseteq N$. Let coalitions S' and T' be coalitions that satisfy $\widetilde{C}(S) = C(S')$ and $\widetilde{C}(T) = C(T')$, where $S' \supseteq S$ and $T' \supseteq T$. Clearly, $C(S') \leq C(T')$ as otherwise, it would be a contradiction to the definitions of $\widetilde{C}(S)$ and S'. Thus, $\widetilde{C}(S) \leq \widetilde{C}(T)$.
4. If the game (N, C) is monotone then it is not possible to reduce the cost of any coalition $S \subseteq N$ by adding to it new players from $N \backslash S$, thus $\widetilde{C}(S) = C(S)$. ■

The monotonicity of the auxiliary game $(N, \widetilde{C})$ implies that it is a nonnegative game as $0 = \widetilde{C}(\emptyset) \leq \widetilde{C}(S)$ for any coalition $S \subseteq N$.

Definition 2 The game $(N, \widetilde{C})$ of a given game (N, C) is called the auxiliary game of (N, C). A coalition $T \subseteq N$ is said to be minimal for coalition $S \subseteq T$, if and only if $\widetilde{C}(S) = C(T)$.

A minimal coalition of a given coalition is not necessarily unique. The maximal size minimal coalition of any given coalition in the consolidation game of parallel $M/M/1$ queueing systems analyzed in Anily and Haviv (2010), is proved to be unique. In Section 4, the uniqueness of the maximal size minimal coalition is generalized to any centralizing aggregation game.

A notion related to the auxiliary game is proposed in Drechsel and Kimms (2010), where the *subcoalition-perfect core* for any cooperative game, is defined. The subcoalition-perfect core of a nonnegative balanced game is, actually, the core of the auxiliary game, defined in Anily and Haviv (2010)) and used here. Note that the subcoalition-perfect core is not the core of the auxiliary game $(N, \widetilde{C})$, if the characteristic function $C: 2^N \rightarrow \mathfrak{R}$ assumes negative values as $\widetilde{C}(\emptyset)$ would be negative, by definition of $\widetilde{C}$, contradicting the definition of a game.

The next theorem demonstrates some special features of auxiliary games that make them helpful in proving total balancedness of certain games.

Theorem 2 *If the auxiliary game $(N, \widetilde{C})$ of a given game (N, C) is (totally) balanced, then*

1. *The game (N, C) is also (totally) balanced.*
2. *The core of the game $(N, \widetilde{C})$ coincides with the non-negative core of the game (N, C).*

Proof

1. If the game $(N, \widetilde{C})$ is balanced, then its core is nonempty. Let $\mathbf{x} = (x_1, \ldots, x_n)$ be a core allocation of $(N, \widetilde{C})$. We will show that this cost allocation is also a core allocation of the game (N, C). For that sake we prove that the efficiency and the stand-alone conditions hold. Regarding the efficiency, note that $\sum_{i=1}^{n} x_i = \widetilde{C}(N) = C(N)$ by using the second item of Theorem 1. For the stand-alone conditions, consider any coalition $S \subset N$: $\sum_{i \in S} x_i \leq \widetilde{C}(S) \leq C(S)$, where the first inequality follows from the fact that the vector $\mathbf{x}$ is in the core of the game $(N, \widetilde{C})$. The second inequality follows from the first item in Theorem 1. If $(N, \widetilde{C})$ is totally balanced, then the same proof applies for any subgame $(S, \widetilde{C})$, $\emptyset \subset S \subset N$, proving that (N, C) is totally balanced.
2. The nonnegativity of the core of the game $(N, \widetilde{C})$ follows from the third item in Theorem 1, and Lemma 1 in Drechsel and Kimms (2010), which prove that the core of a monotone game, if nonempty, is nonnegative. The rest of the proof follows from Theorem 1 in Drechsel and Kimms (2010). ■

Theorem 2 provides a tighter formulation of the nonnegative core of a nonmonotone game (N, C), but its help in identifying core cost allocations of a (totally) balanced game (N, C), is not clear. According to Drechsel and Kimms (2010), the ellipsoid method might provide an element in the core in many applications. Our goal is to provide sufficient conditions under which the auxiliary game is concave, as is the case with the consolidation game of parallel $M/M/1$ queueing systems in Anily and Haviv (2010). Under concavity, the core of the auxiliary game is a polyhedron where each of its $n!$ nonnegative extreme points is associated with a certain order of the n players in N, so that the ith entry of the extreme point, for $1 \leq i \leq n$, is the marginal cost of adding the ith player to the $i-1$ players preceding her (see Shapley, 1971); that is, under concavity of the auxiliary game, the whole nonnegative core of the game (N, C) is identified. The auxiliary game is not helpful in identifying core cost allocations with negative entries, where some players get paid by others in order to persuade them to join the grand-coalition.

3 | CENTRALIZING AGGREGATION GAMES

This section lays the foundations for the structure of games that we focus on and demonstrates them by two examples. *Regular games* are as introduced in Anily and Haviv (2014) and described next: a game (N, C) is *regular*, if each of its n players owns some quantities of an integer number $\kappa \geq 1$ of resources indexed by $\ell = 1 \ldots, \kappa$. Thus, each player $i \in N$ is associated with a vector $y^i \in D \subseteq \mathfrak{R}^\kappa$, called its *vector of properties*, where y^i_ℓ specifies the amount of resource ℓ, $1 \leq \ell \leq \kappa$, that is initially owned by player i. The set D is called the *permissible set*. In a regular game, the cost $C(S)$ induced by any coalition $S \subseteq N$ of size $s = |S| \geq 1$, is a symmetric function of the s vectors of properties owned by its players. More specifically, $C(S)$ can be represented by a symmetric function C_s of the s vectors of properties of S, that is, $C_s : D^s \to \mathfrak{R}$, where the function value is independent of the order of the input vectors. Let $C_0 \equiv 0$ represent the cost of the empty coalition, and for any singleton coalition $C(\{i\}) = C_1(y^i)$ for $i \in N$. Any regular game is associated with a vector $y^0 \in D$, called the *null vector of properties*, whose cost $C_1(y^0) = 0$. The role of y^0 is to guarantee that the sequence $\{C_s\}_{s \geq 0}$ is consistent by linking any two consecutive functions by $C_k(y^1, \ldots, y^k) = C_{k+1}(y^0, y^1, \ldots, y^k)$, for any collection $(y^1, \ldots, y^k) \in D^k$.

The class of regular games is quite large and it contains many well-known games in economics, operations and service management, graph theory etc. A regular game is easily extendable to any set of players once that each player is associated with a vector of properties. As a consequence, it is possible to duplicate players as shown in Definition 1. Let $\mathbf{y^{(m)}}$ denote a sequence of m vectors of properties $y^1, \ldots, y^m$ in D. The next definition provides a formal description of regular games:

Definition 3 An infinite sequence of symmetric functions $\{C_m\}_{m \geq 0}$ is said to be Infinite Increasing Input-Size Symmetric Sequence (IIISSS) of functions for a permissible set D in $\mathfrak{R}^\kappa$, where $\kappa \geq 1$ is integer, if (i) $C_0 \equiv 0$ and for any $m \geq 1$, $C_m : D^m \to \mathfrak{R}$, and (ii) there exists a null vector $y^0 \in D$ such that $C_1(y^0) = 0$ and for any given sequence of $m-1$ vectors of properties $\mathbf{y}^{(m-1)} \in D^{m-1}$, $C_{m-1}(\mathbf{y}^{[m-1]}) = C_m(\mathbf{y}^{(m-1)}, y^0)$. Moreover, any game (N, C) with $|N| \geq 1$ players, where each player $i \in N$ is associated with a vector of properties $y^i \in D$, such that for any $S \subseteq N$, $C(S) = C_{|S|}(y^i|_{i \in S})$, is called a regular game.

An aggregation function $g^{(m)}$ maps the m-fold, $m \geq 1$, cartesian product of the set D into itself, so that $g^{(1)}(y) = y$, for any $y \in D$, and $g^{(m)} : D^m \to D$ aggregates m vectors of properties into one, that is, $g^{(m)}(\mathbf{y^{(m)}}) \in D$ for any $\mathbf{y^{(m)}} \in D^m$, $m > 1$. Next, aggregation games are defined.

Definition 4 Let (N, C) be a regular game that is associated with the IIISSS of functions $C_\ell : D^\ell \to \mathfrak{R}$, where $\ell \geq 0$, and D is a permissible set. If (i) there exists an aggregation function $g^{(m)} : D^m \to D$, such that $C_m(\mathbf{y}^{(m)}) = C_1(g^{(m)}(\mathbf{y}^{(m)}))$, for any $\mathbf{y}^{(m)} \in D^m$, $m \geq 1$, and (ii) the function $C_1 \circ g^{(2)} : D^2 \to \mathfrak{R}$ is reflexive, symmetric, and it satisfies the commutative and the associative laws, then the game is an aggregation game.

According to Definition 4, an aggregation game (N, C), which is associated with an IIISSS of functions $\{C_k\}_{k \geq 0}$, is fully characterized by its aggregation function $g^{(2)} : D^2 \to D$ and the cost function $C_1 : D \to \mathfrak{R}$. An aggregation game with $D = \mathfrak{R}^+_0$, $y^0 = 0$, and $g^{(2)}(x, y) = x + y$, is considered in Özen, Reiman, and Wang (2011). The paper proves that the game is totally balanced if and only if the function C_1 is elastic.

Next, centralizing aggregation games are defined:

Definition 5 An IIISSS of functions $\{C_k\}_{k \geq 0}$ (N, C) is said to be centralizing if for any $m \geq 2$, $\mathbf{y}^{(m)} \in D^m$ with $C_1(y^1) \leq \ldots \leq C_1(y^m)$, the following properties hold (i) $C_1(y^1) \leq C_m(\mathbf{y}^{(m)}) \leq C_1(y^m)$, and (ii) $C_{m+1}(\mathbf{y}^{[m]}, z)$ is strictly increasing in $C_1(z)$, for any $z \in D$. An aggregation game (N, C) that is associated with a centralizing IIISSS of functions is called a centralizing aggregation game.

Theorem 3 provides an alternative definition for a centralizing aggregation game in terms of C_1 and the aggregation function $g^{(2)}$. The proof is by induction using the properties of aggregation functions and Definition 5.

Theorem 3 *An aggregation game defined by the aggregation function* $g^{(2)} : D^2 \to D$ *and the function* $C_1 : D \to \mathfrak{R}$ *is a centralizing aggregation game if and only if.*

- $C_1(y) \leq C_1(z)$ *implies that* $C_1(y) \leq C_1(g^{(2)}(y, z)) \leq C_1(z)$.
- $C_1(g^{(2)}(y, z))$ *is strictly increasing in* $C_1(z)$ *for any* $y, z \in D$.

Next, we present two examples of centralizing aggregation games where each player is associated with a parameter that denotes its cost, and the characteristic function value of a coalition is either the arithmetic or the geometric mean cost of the coalition's members. The games are easily verified to be centralizing aggregation games by Theorem 3. Let $N = \{1, 2, \ldots\}$ be the set of natural numbers, and $N_0 = N \cup \{0\}$.

- The arithmetic mean game (presented also in Anily & Haviv, 2014): each player $i \in N$ is associated with a cost $\alpha_i \in \mathfrak{R}$. The cost of a coalition is the average cost of the coalition's members. In order to define the game as an aggregation game let $\kappa = 2$, $y^0 = (0, 0)$, $D = \{(0, 0)\} \cup \{(x, \ell) : x \in \mathfrak{R}, \ell \in N\}$. Each player is associated with a vector of properties $(a, 1)$ where $a \in \mathfrak{R}$ is her cost. In addition, let the aggregation function $g^{(2)} : D^2 \to D$ be defined as $g^{(2)}((x_1, k_1), (x_2, k_2)) = (x_1 + x_2, k_1 + k_2)$. Thus, each group of k players is associated with a vector of properties of the form (x, k) where x is the sum of the costs of its players. Let $C_1(y^0) = 0$ and otherwise, for any $(x, k) \in D$, $(x, k) \neq y^0$, let $C_1((x, k)) = \frac{x}{k}$. Given a collection of $m \geq 1$ vectors of properties $\mathbf{y^{(m)}} \in D^m$, with $y^i = (x_i, k_i) \in \mathbf{y^{(m)}}$, $i = 1, \ldots, m$, let $C_m(\mathbf{y^{(m)}}) = C_1(g^{(m)}(\mathbf{y^{(m)}}))$. If $g^{(m)}(\mathbf{y^{(m)}}) \neq y^0$, then $C_1(g^{(m)}(\mathbf{y^{(m)}})) = \sum_{i=1}^{m} x_i / \sum_{i=1}^{m} k_i$, and otherwise $C_1(g^{(m)}(\mathbf{y^{(m)}})) = 0$. It is easy to see that this game satisfies the conditions of Theorem 3, and therefore it is a centralizing aggregation game. We conclude this section by proving a theorem from which we deduce that the nonnegative version of this game, that is, the case where the permissible set $D = \{(0, 0)\} \cup \{(x, \ell) : x \geq 0, \ell \in N\}$, is subadditive, and then showing by an instance that the arithmetic game with negative entries is not necessarily subadditive. The analysis of a scheduling game in subsection 1.2 implies that the nonnegative version of the arithmetic mean game is totally balanced.
- The geometric mean game: each player $i \in N$ is associated with a cost $\alpha_i > 0$. The cost of a coalition is the geometric mean of the costs of its members. For example, the cost of a coalition of three players is the cube root of the product of the costs that are associated with the players. Let $\kappa = 2$. Each player is associated with a vector of properties $(a, 1)$ where a is her cost. Unlike the arithmetic mean game, here, the null vector of properties is $y^0 = (1, 0)$ and $D = \{(1, 0)\} \cup \{(x, \ell) : x > 0, \ell \in N\}$. Let the aggregation function $g^{(2)}((x_1, k_1), (x_2, k_2)) = (x_1 x_2, k_1 + k_2)$. Each group of k players is associated with a vector of properties of the form (x, k) where x is the product of the costs of the group's players, and k is the number of players in the group. Let $C_1(y^0) = 0$ and otherwise, for any $(x, k) \in D$, let $C_1((x, k)) = x^{1/k}$. The corresponding IIISSS of functions for a given collection of $m \geq 1$ vectors of properties $\mathbf{y^{(m)}} \in D^m$, where $y^i = (x_i, k_i) \in \mathbf{y^{(m)}}$, $i = 1, \ldots, m$, is given by $C_m(\mathbf{y^{(m)}}) = C_1(g^{(m)}(\mathbf{y^{(m)}}))$. If $g^{(m)}(\mathbf{y^{(m)}}) \neq y^0$, then $C_1(g^{(m)}(\mathbf{y^{(m)}})) = (\Pi_{i=1}^{m} x_i)^{(\Sigma_{i=1}^{m} k_i)^{-1}}$, that is, it is the geometric mean of the costs of the vectors of properties in D, and otherwise $C_1(g^{(m)}(\mathbf{y^{(m)}})) = 0$. As the conditions of Theorem 3 are satisfied, the geometric mean game presented here is a centralizing aggregation game.

As shown in the above two examples, the choice of the null vector of properties y^0 depends on the definition of the game, as the second requirement in Definition 3 must be satisfied. In Section 2 we pose an open question regarding the total balancedness of a large family of centralizing aggregation games that contains as private cases the arithmetic mean and the geometric mean games. In this family, each player is associated with a cost, and the characteristic function of a coalition is a generalized mean of the costs of its members. Note that taking the characteristic function to be the maximum of a set of numbers is one extreme special case of a generalized mean. Its associated game, which is well known as the *airport game* (see Littlechild & Owen, 1973), can easily be presented as an aggregation game, but it is not a centralizing game as it does not satisfy the second item of Theorem 3, that is, $C_1(g^{(2)}(y, z))$ for $y, z \in D$, is not strictly increasing in $C_1(z)$, as required. Yet, the airport game is easily shown to be concave and therefore it is totally balanced, see Condition 1.

Theorem 4 *A centralizing aggregation game (N, C) is homogenous of degree 0 and non-monotone. If, in addition, the game (N, C) is non-negative, that is,* $C : 2^N \to \mathfrak{R}_0^+$, *then the game is sub-additive.*

Proof Consider any two players i, $j \in N$ and their vectors of properties y^i, $y^j \in D$ such that $C_1(y^i) < C_1(y^j)$. By Theorem 3, $C_1(y^i) = C_1(g^{(2)}(y^i, y^i)) < C_1(g^{(2)}(y^i, y^j)) < C_1(g^{(2)}(y^j, y^j)) = C_1(y^j)$, proving that adding a player to a coalition may increase or decrease the cost of the coalition, that is, the game is nonmonotone. In order to show that the game is homogenous of degree 0, (see Definition 1), take m copies of $S \subseteq N$, denoted by $S^1, \ldots, S^m$. Thus, $C(S^1) = \ldots = C(S^m) = C(S)$. Let y^S ($y^{S^k}, k = 1, \ldots, m$) be the vector of properties in D that is obtained by aggregating the vectors of properties of S (S^k, $k = 1, \ldots, m$). Thus, $C(S) = C_1(y^S) = C_1(y^{S^k})$ for $k = 1, \ldots, m$. The centralizing property implies that $C(S) = C_1(y^{S^1}) \leq C_m(g^{(m)}(y^{S^1}, \ldots, y^{S^m})) \leq C_1(y^{S^m}) = C(S)$, proving that the cost of a set in which each player of coalition S is cloned m times, is the same as the cost of S. Finally, consider any two disjoint coalitions $S, T \subset N$. Without loss of generality (w.l.o.g) suppose that $C(S) \leq C(T)$. Let y^S (y^T) be the vectors of properties in D obtained by aggregating the vectors of S (T), respectively, that is, $C(S) = C_1(y^S)$

and $C(T)=C_1(y^T)$. In view of the centralizing property, $C(S)=C_1(y^S)\le C_1(g^{(2)}(y^S, y^T))\le C_1(y^T)=C(T)$, where $C_1(g^{(2)}(y^S, y^T))=C(S\cup T)$. As the game is nonnegative, $C(S)\ge 0$, implying that the game is subadditive as $C(S\cup T)\le C(S)+C(T)$. ∎

Subaddivity is a necessary condition for a game to have a nonempty core. The second item of Theorem 4 states that nonnegativity is a sufficient condition for a centralizing aggregation game to be subadditive. The nonnegativity is essential as otherwise a centralizing aggregation game might not be subadditive, as the following example demonstrates: consider the arithmetic mean game that has been shown in this section to be a centralizing aggregation game, where $N=\{1, 2, 3\}$, $a_1=-1$, $a_2=a_3=1$, $S=\{1\}$ and $T=\{2, 3\}$, implying that the game is not subadditive as $1/3=C(S\cup T)>C(S)+C(T)=-1+1=0$. From here on we focus on nonnegative regular games.

4 | THE AUXILIARY GAME OF A CENTRALIZING AGGREGATION GAME

In the sequel we prove further properties that the auxiliary game of a centralizing aggregation game, satisfies. Consider a game (N, C), and its auxiliary game $(N, \widetilde{C})$ as defined in Section 2. Recall that a minimal coalition S^{prime} of any coalition $S\subseteq N$, is a minimum cost superset of S in N (see Definition 2). For any coalition $S\subseteq N$, let $\widetilde{S}$ be one of its maximal size minimal coalitions. Regularity of (N, C) implies that each player $i\in N$ is associated with a vector of properties $y^i\in D\subset\mathfrak{R}^\kappa$ and its cost, if the player acts individually, is $C_1(y^i)\in\mathfrak{R}_0^+$. W.l.o.g., index the players in a nondecreasing order of their costs, that is, $C_1(y^1)\le\ldots\le C_1(y^n)$.

Theorem 5 *For any coalition $S\subset N$ of a centralizing aggregation game (N, C), there exists a unique maximal size minimal set $\widetilde{S}$. If $S=\{1, \ldots, k\}$ and $C_1(y^k)<C_1(y^{k+1})$, then $\widetilde{S}=S$. Otherwise, $\widetilde{S}=\{1,\ldots,j\}\cup S$ for some $j\in N\backslash S$ where $C_1(y^j)\le C(\widetilde{S})\le C(\widetilde{S}\backslash\{j\})$. If $\widetilde{S}\ne N$ then $C(\widetilde{S})<C_1(y^p)$, where $p=\min\{i: i>j,\ i\in N\backslash\widetilde{S}\}$. Moreover, if $\emptyset\subset S\subset T\subseteq N$, then $\widetilde{S}\subseteq\widetilde{T}$.*

Proof Consider first the case where $S=\{1,\ldots,k\}$ and $C_1(y^k)<C_1(y^{k+1})$. Let $y^S=g^{(k)}(y^1,\ldots,y^k)$. $C(S)=C_1(y^S)\le C_1(y^k)$, where the last inequality follows from centralizing property of the IIISSS of functions. Thus, by Theorem 3, $C_1(y^S)<C_1(g^{(2)}(y^S, y^\ell)$ for any $\ell\ge k+1$, implying that S is the unique maximal size minimal set of S. Otherwise, proving that $\widetilde{S}$ is of the form $\{1,\ldots,j\}\cup S$ for some $j\in N\backslash S$ is equivalent to proving that there does not exist any pair of players $1\le\ell<k\le n$ such that $\{\ell, k\}\subset N\backslash S$, $\ell\notin\widetilde{S}$ and $k\in\widetilde{S}$. Suppose by contradiction that such a pair $\{\ell, k\}$ existed. Let $\widetilde{S}_k=\widetilde{S}\backslash\{k\}$ and z (w) be the vector of properties that is obtained by aggregating the vectors of properties of $\widetilde{S}$ $(\widetilde{S}_k)$. Thus, $C(\widetilde{S})=C_1(z)=C_1(g^{(2)}(w,y^k))$. As the players are indexed in a nondecreasing order of their costs, $\ell<k$, and the game is a centralizing aggregation game, see Section 3, it follows that $C_1(y^\ell)\le C_2(y^\ell, y^k)\le C_1(y^k)$, implying that $C_1(g^{(2)}(w, g^{(2)}(y^\ell y^k)))\le C_1(g^{(2)}(w,y^k))=C(\widetilde{S})$, which means that adding player ℓ to $\widetilde{S}$ results in a coalition that strictly contains $\widetilde{S}$ and its cost is at most $C(\widetilde{S})$, contradicting the assumption that $\widetilde{S}$ is a maximal size minimal coalition of S. ∎

Next we prove the inequalities stated in the theorem: suppose by contradiction that $C_1(y^j)>C(\widetilde{S}\backslash\{j\})$. Using this assumption with the properties of centralizing aggregation games, implies that coalition $\widetilde{S}\backslash\{j\}$, $j\in N\backslash S$, costs strictly less than coalition $\widetilde{S}$, contradicting the fact that $\widetilde{S}$ is a minimal coalition for S. Thus, $C_1(y^j)\le C(\widetilde{S}\backslash\{j\})$ implying that $C_1(y^j)\le C(\widetilde{S})\le C(\widetilde{S}\backslash\{j\})$. Finally, if $\widetilde{S}\ne N$, then player p, as defined in the theorem, is the lowest indexed player of N that is not a member of $\widetilde{S}$. If $C_1(y^p)\le C(\widetilde{S})$ then $C(\widetilde{S}\cup\{p\})\le C(\widetilde{S})$ contradicting the fact that $\widetilde{S}$ is a maximal size minimal coalition of S, thus $C_1(y^p)>C(\widetilde{S})$. Finally, we prove the last assertion stating that if one set is contained in another set, then the maximal minimal set of the first is contained in the maximal minimal set of the other. According to Theorem 1, the game $(N,\widetilde{C})$ is monotone. Thus, if $\widetilde{S}=N$ then $\widetilde{C}(T)\ge\widetilde{C}(S)=C(N)$, that is, $\widetilde{T}=N$. Therefore, if $\widetilde{S}=N$ or $\widetilde{S}=S$ the inequality $\widetilde{S}\subseteq\widetilde{T}$ holds trivially. Next we prove the case $\widetilde{S}\notin\{S,N\}$: Let b_S be the highest indexed player added to S toward the formation of $\widetilde{S}$ while c_S is the lowest indexed player left out. If $b_S=0$ then $\widetilde{S}=S$ and if $c_s=n+1$, then $\widetilde{S}=N$. Thus, it remains to focus on the case that $b_S>0$ and $c_S\le n$. Suppose by contradiction that $\widetilde{S}$ is not a subset of $\widetilde{T}$. In such a case there exists a player $k\le b_S$ satisfying $k\in\widetilde{S}$, but $k\notin\widetilde{T}$. This means that $b_T<c_T\le k\le b_S$, which implies that $C_1(y^{b_T})\le\widetilde{C}(T)<C_1(y^{c_T})\le C_1(y^k)\le C_1(y^{b_S})\le\widetilde{C}(S)$, contradicting the monotonicity of the game $(N,\widetilde{C})$. Therefore, $\widetilde{S}\subseteq\widetilde{T}$.

Next we generalize the construction algorithm introduced in Anily and Haviv (2010) to any centralizing aggregation game. For a given such game (N, C), and for any coalition $S\subset N$, the construction algorithm returns both the maximal size minimal set $\widetilde{S}$ and the cost $\widetilde{C}(S)$, $S\ne N$. The algorithm is a greedy-type algorithm: for any coalition $S\subseteq N$, it starts with $\widetilde{S}=S$ and gradually adds to $\widetilde{S}$ the lowest indexed player that is still in $N\backslash\widetilde{S}$, as long as the cost of $\widetilde{S}$ does not increase by adding the new player. As in the proof of Theorem 5, let b_S be the highest indexed player added to S toward the formation of $\widetilde{S}$ while c_S is the lowest indexed player left out.

The Auxiliary Game Construction Algorithm for a centralizing aggregation game:

input: A game (N, C) and S, $\emptyset \subseteq S \subsetneq N$.
output: $\widetilde{C}(S)$ and $\widetilde{S}$.
Step 0: Initialization: $\widetilde{S} = S$; $S^c = N \backslash S$, $c_S = \min S^c$ and $b_S = 0$.
Step 1: While $C(\widetilde{S}) \geq C_1(\{c_S\})$ let $\widetilde{S} \leftarrow \widetilde{S} \cup \{c_S\}$; $b_S = c_S$; $S^c \leftarrow S^c \backslash \{c_S\}$; If $S^c \neq \emptyset$ then $c_S = \min S^c$. Otherwise, $c_S = n + 1$ and goto Step 2. Endwhile.
Step 2: Let $\widetilde{C}(S) = C(\widetilde{S})$. Return $\widetilde{S}$, $\widetilde{C}(S)$, b_S, and c_S.
End of Algorithm.

5 | THE MAIN THEOREM

In this section we provide a sufficient condition under which the nonnegative core of a centralizing aggregation game is fully characterized. For this sake, we introduce a decreasing differences condition on the centralizing aggregation function $C_1 {}^\circ g^{(2)} : D^2 \to \mathfrak{R}$. Recall that according to Theorem 3 on centralizing aggregation games, for any three vectors of properties w, x and y: $C_1(w) \geq C_1(y)$ implies that $C_1(g^{(2)}(x, w)) - C_1(g^{(2)}(x, y)) \geq 0$.

Definition 6 The characteristic function of a centralizing aggregation game given by $C_1 {}^\circ g^{(2)} : D^2 \to \mathfrak{R}$ where $C_1 : D \to \mathfrak{R}$, and $g : D^2 \to D$, for $D \subseteq \mathfrak{R}^\kappa$, $\kappa \geq 1$, is said to have decreasing differences if $C_1(g^{(2)}(x, w)) - C_1(g^{(2)}(x, y)) \leq C_1(w) - C_1(y)$ for any vectors of properties x, w, y $\in$ D that satisfy $C_1(x) \geq C_1(w) \geq C_1(y)$.

The decreasing differences property has some similarity to the concavity of real functions $f : \mathfrak{R} \to \mathfrak{R}$, that satisfy $f(w+\Delta) - f(y+\Delta) \leq f(w) - f(y)$ for any $\Delta \geq 0$ and $w \geq y$. Note that the condition $C_1(x) \geq C_1(w) \geq C_1(y)$ for centralizing aggregation games, plays the role of the requirement $\Delta \geq 0$ in concavity, as it ensures that $C_1(g^{(2)}(x, w)) \geq C_1(w)$ and $C_1(g^{(2)}(x, y)) \geq C_1(y)$. Yet, as the following example shows, the decreasing differences property is a much weaker condition than concavity, as the function C_1 is not necessarily continuous, where concavity implies that the function is continuous: consider the arithmetic mean game presented in Section 3, but now let the permissible set be $D = \{(q, k) : q \in Q_0^+ \text{ and } k \in N\}$, where Q_0^+ is the set of nonnegative rational numbers. This noncontinuous version of the game, as its continuous nonnegative version, satisfy the decreasing differences property.

In the proof of the main theorem, we use an alternative condition for a game to be concave (see Shapley, 1971) that is equivalent to Condition 1 in Section 1. Let $S \cup \{\ell\} = S_\ell$ for any coalition $S \subset N \backslash \{\ell\}$:

Property 1 A cooperative game (N, C) is concave if and only if it satisfies the following property for any $S \subset T \subset N$ and for any $\ell \in N \backslash T$:

$$C(T_\ell) - C(T) \leq C(S_\ell) - C(S).$$

Theorem 6 *Consider a centralizing aggregation game (N, C) defined by C_1 and the aggregation function $g^{(2)} : D^2 \to D$, whose auxiliary game is $(N, \widetilde{C})$. If the function $C_1 {}^\circ g^{(2)} : D^2 \to \mathfrak{R}^+$ satisfies the decreasing differences property then the auxiliary game $(N, \widetilde{C})$ is concave.*

Proof Theorem 4 implies that the game (N, C) is subadditive and nonmonotone, and by Theorem 1, the auxiliary game $(N, \widetilde{C})$ is monotone. The concavity of $(N, \widetilde{C})$ is proved by using Property 1, that is, by showing that $\widetilde{C}(T_\ell) - \widetilde{C}(T) \leq \widetilde{C}(S_\ell) - \widetilde{C}(S)$ for any $S \subset T \subset T_\ell \subseteq N$. The monotonicity of $\widetilde{C}$ implies that both sides of this inequality are nonnegative. Moreover, if $\ell \in \widetilde{T}$ then $\widetilde{C}(T_\ell) - \widetilde{C}(T) = 0$ and the proof is trivial. Thus, it is left to consider the case that $\widetilde{T}$ is a maximal size minimal set of T and $\ell \notin \widetilde{T}$. Next, we prove the following three inequalities: ■

1. The left-hand side of the inequality satisfies
$$\widetilde{C}(T_\ell) - \widetilde{C}(T) \leq C(\widetilde{T} \cup \{\ell\}) - C(\widetilde{T}).$$
2. The right-hand side of the inequality satisfies
$$\widetilde{C}(S_\ell) - \widetilde{C}(S) \geq C(\widetilde{S_\ell}) - C(\widetilde{S_\ell} \backslash \{\ell\}).$$
3. Finally,
$$C(\widetilde{T} \cup \{\ell\}) - C(\widetilde{T}) \leq C(\widetilde{S_\ell}) - C(\widetilde{S_\ell} \backslash \{\ell\}).$$

1. Note that $\widetilde{C}(T) = C(\widetilde{T})$ by definition, and $\widetilde{C}(T_\ell) = C(\widetilde{T_\ell}) \leq C(\widetilde{T} \cup \{\ell\})$. The inequality holds as the coalition $\widetilde{T} \cup \{\ell\}$ contains T_ℓ but it is not necessarily one of its minimal coalitions, where $\widetilde{T_\ell}$ is.
2. Note that $\widetilde{C}(S_\ell) = C(\widetilde{S_\ell})$ by definition, and $\widetilde{C}(S) \leq C(\widetilde{S_\ell} \backslash \{\ell\})$ as coalition $\widetilde{S_\ell} \backslash \{\ell\}$ contains coalition S but is not necessarily a minimal coalition of S.
3. In order to prove the third item define (a) u^1 as the vector of properties obtained by aggregating all the vectors of properties of $\widetilde{S_\ell} \backslash \{\ell\}$ by using repeatedly the aggregation function g, (b) u^2 as the vector of properties obtained by aggregating all the vectors of properties of $\widetilde{T}$ by using repeatedly the aggregation function g, and (c) $z = y^\ell$ as the vector of properties of player ℓ. Recall the assumption that $\ell \notin \widetilde{T}$ and $\widetilde{T}$ is a maximal size minimal set of T, implying by Theorem 5 that

the strong inequality $C(\widetilde{T}) < C(\{\ell\})$ holds. Therefore, $C_1(u^2) < C_1(z)$. As $S_\ell \subset \widetilde{T} \cup \{\ell\}$, $\widetilde{C}(S_\ell) \leq \widetilde{C}(\widetilde{T} \cup \{\ell\}) \leq C(\widetilde{T} \cup \{\ell\})$, where the first inequality follows from the monotonicity of the game $(N, \widetilde{C})$ and the second inequality follows from the fact that $\widetilde{T} \cup \{\ell\}$ is possibly, but not necessarily, a minimal coalition for itself. By definition, $C(\widetilde{S_\ell}) = C_1(g^{(2)}(u^1, z))$ and $C(\widetilde{T} \cup \{\ell\}) = C_1(g^{(2)}(u^2, z))$. Thus, inequality $C_1(g^{(2)}(u^1,z)) \leq C_1(g^{(2)}(u^2,z))$ follows. By the second subitem of the last item of Definition 3 on centralizing functions, also the inequality $C_1(u^1) \leq C_1(u^2)$ holds. It remains to show that $C_1(g^{(2)}(u^2, z)) - C_1(u^2) \leq C_1(g^{(2)}(u^1, z)) - C_1(u^1)$, or equivalently, that $C_1(g^{(2)}(u^2, z)) - C_1(g^{(2)}(u^1,z)) \leq C_1(u^2) - C_1(u^1)$, which follows from the decreasing differences property of the composition $C_1 \circ g^{(2)} : D^2 \to \mathfrak{R}_0^+$, completing the proof.

The conditions of Theorem 6 do not imply that the game (N, C) is concave. In fact, none of the games presented in Section 1 is concave.

6 | EXAMPLES

A few examples of cooperative games in the areas of operations research and operations management are presented in this section. The first is the queueing model that triggered this research, the second is a scheduling game and the third is a scheduling game associated with the Economic Lot Scheduling Problem (ELSP). We start by presenting a simple property that refers to ratios of nonnegative real numbers.

Property 2 Let $a_i, a_j \geq 0$ and $b_i, b_j > 0$. $\frac{a_i}{b_i} \leq \frac{a_j}{b_j}$ if and only if $\frac{a_i}{b_i} \leq \frac{a_i+a_j}{b_i+b_j} \leq \frac{a_j}{b_j}$. Moreover, $\frac{a_i}{b_i} < \frac{a_j}{b_j}$, if and only if $\frac{a_i}{b_i} < \frac{a_i+a_j}{b_i+b_j} < \frac{a_j}{b_j}$.

6.1 | The consolidation game of parallel *M*/*M*/1 queueing systems

In the model presented in Anily and Haviv (2010) and mentioned in Section 1, servers cooperate in order to minimize the long-run total congestion cost measured by the number of customers in the system. When servers cooperate, they form a single "super-server" whose service rate is the sum of the individual service rates, and its stream of arrivals is the union of the respective streams of arrivals. More precisely, let $N = \{1, \ldots, n\}$ be a set of n *M*/*M*/1 queueing systems. Queueing system i is associated with an exponential service rate μ_i and a Poisson arrival rate λ_i, $\lambda_i < \mu_i$, $i \in N$. Cooperation of a set $S \subseteq N$ results in a single *M*/*M*/1 queue with capacity $\mu(S) = \sum_{i \in S} \mu_i$, and arrival rate $\lambda(S) = \sum_{i \in S} \lambda_i$. The congestion of coalition $S \subseteq N$ is given by

$$C(S) = \frac{\lambda(S)}{\mu(S) - \lambda(S)}.$$

The game is shown to be nonmonotone and nonconcave. The characteristic function of its auxiliary game is proved in Anily and Haviv (2010) to be concave by a long and tedious proof tailored to this specific game. Theorem 6 in Section 5 proposes an alternative sufficient condition for a game to be concave. Applying this condition on the auxiliary game, and then invoking Theorem 2, imply that the original game is totally balanced and the nonnegative part of its core coincides with the core of the auxiliary game.

Next, the game is presented as a centralizing aggregation game. Associate with each system $i \in N$ a vector of properties of size $\kappa = 2$, where its first entry is λ_i, and the second entry is $\delta_i = \mu_i - \lambda_i$. In addition, let, $y^0 = (0, 0)$, and $D = \{0,0\} \cup \{(\lambda\delta) | \lambda \geq 0, \delta > 0\} \subset (\mathfrak{R}_0^+)^2$. Let $C_1(y^0) = 0$, and for $(\lambda, \delta) \in D \backslash \{0, 0)\}$, $C_1((\lambda, \delta)) = \frac{\lambda}{\delta}$. The aggregation function g that combines two vectors of properties in D into one is $g^{(2)}((\lambda_1, \delta_1), (\lambda_2, \delta_2)) = (\lambda_1 + \lambda_2, \delta_1 + \delta_2)$. Consider m vectors of properties $y^1, \ldots, y^m$ in D where $y^i = (\lambda_i, \delta_i)$ for $i = 1 \ldots m$. Thus, $g^{(m)}(y^1, \ldots, y^m) = \left(\sum_{i=1}^m \lambda_i, \sum_{i=1}^m \delta_i\right)$, and $C_m(y^1, \ldots, y^m) = C_1(g^{(m)}(y^1, \ldots, y^m))$. If $g^{(m)}(y^1, \ldots, y^m) \neq y^0$, then $C_1(g^{(m)}(y^1, \ldots, y^m)) = \sum_{i=1}^m \lambda_i / \sum_{i=1}^m \delta_i$, and otherwise it is zero. According to Property 2 and Theorem 3, the game is a centralizing aggregation game.

In order to conclude the simpler alternative analysis of the game by Theorems 6 and 2, it remains to show that $C_1 \circ g^{(2)}$ satisfies the decreasing differences property, see Definition 6. To this end, it is sufficient to show that $C_1(g^{(2)}(u^j, z)) - C_1(g^{(2)}(u^i, z)) \leq C_1(u^j) - C_1(u^i)$, where the vectors $u^i = (a_i, b_i)$, $u^j = (a_j, b_j)$, and $z = (a_k, b_k)$ are in D and they satisfy $C_1(u^i) < C_1(u^j) \leq C_1(z)$. This is equivalent to showing that $\frac{a_j+a_k}{b_j+b_k} - \frac{a_i+a_k}{b_i+b_k} \leq \frac{a_j}{b_j} - \frac{a_i}{b_i}$, which follows from Property 3 below, where $\xi(x)$ is the identity function:

Property 3 Let $\xi : \mathfrak{R}_0^+ \to \mathfrak{R}_0^+$, be a continuous real function which is both increasing and concave. The real numbers $a_i, a_j, a_k \geq 0$, $b_i, b_j, b_k > 0$ satisfy $\frac{a_i}{b_i} < \frac{a_j}{b_j} \leq \frac{a_k}{b_k}$. Then $0 \leq \xi\left(\frac{a_j+a_k}{b_j+b_k}\right) - \xi\left(\frac{a_j}{b_j}\right) < \xi\left(\frac{a_i+a_k}{b_i+b_k}\right) - \xi\left(\frac{a_i}{b_i}\right)$.

Proof *Proving* the property is equivalent to proving that the function $\psi(\alpha, \beta) = \xi\left(\frac{\alpha+\theta}{\beta+\delta}\right) - \xi\left(\frac{\alpha}{\beta}\right)$, where $\theta > 0$, $\delta > 0$ are fixed constants, and $\frac{\alpha}{\beta} < \frac{\theta}{\delta}$, is decreasing in $\frac{\alpha}{\beta}$. In order to prove the property, rewrite the function $\psi(\alpha, \beta)$ as a function π of the two variables β and $\rho = \frac{\alpha}{\beta}$: $\pi(\beta, \rho) = \xi\left(\frac{\rho+\theta\beta^{-1}}{1+\delta\beta^{-1}}\right) - \xi(\rho)$. It remains to verify that the function $\pi(\beta, \rho)$ is decreasing in ρ. This is done by checking the sign of the first partial derivative of π with respect to ρ,

$$\frac{\partial \pi(\beta,\rho)}{\partial \rho} = \left(\frac{1}{1+\delta\beta^{-1}}\right)\frac{\partial \xi\left(\frac{\rho+\theta\beta^{-1}}{1+\delta\beta^{-1}}\right)}{\partial \rho} - \frac{\partial \xi(\rho)}{\partial \rho}. \blacksquare$$

According to Property 2, $\frac{\alpha+\theta}{\beta+\delta} > \frac{\alpha}{\beta}$, thus by using the concavity of the function ξ we deduce that $\frac{\partial \xi\left(\frac{\rho+\theta\beta^{-1}}{1+\delta\beta^{-1}}\right)}{\partial \rho} \le \frac{\partial \xi(\rho)}{\partial \rho}$. In addition, as the function ξ is increasing $\frac{\partial \xi(\rho)}{\partial \rho} \ge 0$. The proof is concluded as:

$$\frac{\partial \pi(\beta,\rho)}{\partial \rho} \le \left(\frac{1}{1+\delta\beta^{-1}} - 1\right)\frac{\partial \xi(\rho)}{\partial \rho} \le 0$$

Note that Property 3 holds for the nonnegative version of the arithmetic mean game, presented in Section 3, by substituting the function ξ by $\xi(x)=x$.

6.2 | Minimizing Makespan with Preemptions

Consider a number of production units $i \in N=\{1, \ldots, n\}$, hereafter called players. Each player $i \in N$ owns $k_i \ge 1$ machines that process a collection of jobs whose total processing time is $p_i \ge 0$, and its longest job is of duration $q_i \in [0, p_i]$. The machines of all players are assumed to be identical in all aspects, including their speed, and they work in parallel. Each player, if acting individually, schedules his jobs on his machines so that their makespan is minimized. The schedules allow for preemptions, but a job cannot be processed simultaneously on different machines. For the optimal solution to the scheduling problem of a player (see Pinedo, 2002, ch. 5): the minimum makespan of a player is the maximum between his longest job and the ratio between his total processing time and the number of machines that he owns. If the makespan is determined by the longest processing time, then it is possible that the optimal schedule on some of the machines includes idle times. Otherwise, all machines are fully utilized during the makespan duration. We propose a cooperative game where players form coalitions by sharing their machines in order to minimize the makespan of their jobs. For any coalition $S \subseteq N$, let $k(S) = \sum_{i \in S} k_i$ be the number of machines owned by coalition S, $q(S)=\max\{q_i : i \in S\}$, be the duration of the longest job in S, and $p(S) = \sum_{i \in S} p_i$, be the total processing time of all jobs in S. Clearly, $p(S) \ge q(S)$. If $q(S)=0$ then also $p(S)=0$. Let (N, C) be the respective game where the characteristic function value $C(S)$ for $S \subseteq N$ denotes the optimal makespan of running the jobs of S. More specifically, $C(S) = \max\left\{q(S), \frac{p(S)}{k(S)}\right\}$. If the players break into m disjoint coalitions $S_1, \ldots, S_m$, such that $N = \cup_{\ell=1}^{m} S_\ell$, then the total cost is $\sum_{\ell=1}^{m} C(S_\ell)$. It is easy to see that this game is subadditive.

The following instance shows that the makespan with preemptions game is not concave: consider three players with the following parameters: $p_1=9$, $q_1=1$, $k_1=1$, $p_2=5$, $q_2=0.5$, $k_2=5$, and $p_3=1$, $q_3=0.01$, $k_3=9$. Let $S \subset T$, where $S=\{1\}$, $T=\{1, 2\}$, and $\ell=3$. Using the notations of Property 1, $S_3=\{1, 3\}$, and $T_3=\{1, 2, 3\}$. Thus, $C(\{1\})=9$, $C(\{1, 2\})=7/3$, $C(\{1, 3\})=1$, and $C(\{1, 2, 3\})=1$. The concavity property does not hold as $C(T_3)-C(T)=1-7/3=-4/3 > C(S_3)-C(S)=1-9=-8$, and therefore, Condition 1 in Section 1 cannot be invoked. Yet, as we show below, the game is totally balanced.

Next, we represent the game as an aggregation game: each player $i \in N$ is initially associated with a vector of properties of size 3, (p_i, q_i, k_i), such that $p_i \ge q_i > 0$ or $p_i = q_i = 0$. Let the null vector of properties $y^0=(0, 0, 0)$, $D=\{(p, q, k)$: $(p \ge q > 0$, or $q=p=0)$, and $k=1, 2 \ldots\}$, and the aggregation function $g^{(2)}((p_1,q_1,k_1),(p_2,q_2,k_2))=(p_1+p_2, \max\{q_1,q_2\}, k_1+k_2)$. Finally, let

$$C_1(p,q,k) = \begin{cases} \max\{q, p/k\} & \text{if} \quad (q,p,k) \ne y^0 \\ 0 & \text{otherwise.} \end{cases}$$

The game (N, C) is not a centralizing game though $C_1(g^{(2)}((p_1,q_1,k_1),(p_2,q_2,k_2))) \in [\min\{C_1(p_i,q_i,k_i) : i=1,2\}, \max\{C_1(p_i,q_i,k_i) : i=1,2\}]$, as it is not strictly increasing in $C_1(p_2, q_2, k_2)$ while (p_1, q_1, k_1) is kept fixed. This can be seen by taking, for example, $(p_1, q_1, k_1)=(5, 3, 2)$ and $(p_2, q_2, k_2)=(6, b, 3)$, implying that $C_1(5, 3, 2)=3$, and for $b \ge 2$, $C_1(6, b, 3)=b$. In particular, for $b \in (2, 3)$ the value of $C_1(6, b, 3)=b$ is strictly increasing in b where the value of the aggregated vector, $C_1(g^{(2)}((5, 3, 2), (6, b, 3)))=C_1((11, 3, 5))=3$, is not, implying that the second item in Theorem 3 is not satisfied, and Theorem 6 cannot be invoked. Thus, we present a different approach, which is based on the next property:

Property 4 Consider a game (N, C), where $C(S)=\max\{U^1(S), U^2(S), \ldots, U^L(S)\}$, $L \ge 2$, for any sequence of set functions $U^1, U^2, \ldots, U^L$ defined on N and any coalition $S \subseteq N$. If each game (N, U^k) for $k=1 \ldots L$, is totally balanced, then, the game (N, C) is also totally balanced. Moreover, let $k^* = \arg\max\{U^k(N) : k=1 \ldots L\}$, then the core of the game (N, U^{k^*}) is a subset of the core of the game (N, C).

Proof Let $\vec{\alpha} = (\alpha_1, \ldots, \alpha_n)$ be a core cost allocation of the game (N, U^{k^*}). It is sufficient to show that $\vec{\alpha}$ is a core cost allocation for the game (N, C). The efficiency property of the cost allocation $\vec{\alpha}$ holds as $\sum_{i=1}^{n} \alpha_i = C(N)$ follows from the choice of k^*. For any proper coalition $S \subset N$, the stand-alone property for (N, C) holds as $\sum_{i \in S} \alpha_i \le U^{k^*}(S) \le \max\{U^k(S) : k = 1 \ldots L\} = C(S)$. $\blacksquare$

Claim 1 The makespan with preemptions game is totally balanced, and a polytope, which is a subset of its nonnegative core, is fully characterized.

Proof The characteristic function of the makespan game with preemptions (N, C) can be presented as $C(S) = \max\{U^1(S), U^2(S)\}$ where $U^1(S) = q(S) = \max\{q_i : i \in S\}$, and $U^2(S) = \frac{p(S)}{k(S)} = \frac{\sum_{i \in S} p_i}{\sum_{i \in S} k_i}$. The game (N, U^1), is the airport game mentioned in Section 3, which is known to be concave (see (Littlechild & Owen, 1973), and therefore, it is totally balanced, and its core is fully characterized. Moreover, the game is monotone, implying that its core is nonnegative. The characteristic function of the game (N, U^2) has the same structure as the one of the game in Subsection 1.1, that is, it is the ratio of the sum of non-negative real entries of the vectors of properties by the sum of positive real entries of the vectors of properties, and it is defined as 0 for the null vector of properties. As discussed in Subsection 1.1, a game with this form of a characteristic function, is a centralizing aggregation game that has the decreasing differences property. Therefore, according to Theorem 6, the game (N, U^2) is totally balanced, and its nonnegative core is fully identified. The total balancedness of the makespan game with preemptions thus follows from Property 4. ■

Finally, a numerical example of a game with three players, is presented: Let $N = \{1, 2, 3\}$ be a set of players with the following parameters: $p_1 = 20$, $q_1 = 8$, $k_1 = 4$, $p_2 = 30$, $q_2 = 3$, $k_2 = 6$, $p_3 = 40$, $q_3 = 5$, and $k_3 = 10$. For each coalition the characteristic function value is computed: Thus, $C(\{1\}) = 8$, $C(\{2\}) = 5$, $C(\{3\}) = 5$, $C(\{1, 2\}) = 8$, $C(\{1, 3\}) = 8$, $C(\{2, 3\}) = 5$, $C(N) = 8$.According to Property 4, as $C(N) = U^1(N) = 8$, the duration of the longest job in N, the core of (N, U^1) is a subset of the core of (N, C). In view of the concavity of the game (N, U^1), its core is the polytope defined by the 3! extreme points, where each extreme point is the marginal contribution vector for a certain order of the players (see Shapley, 1971). The characteristic function values of the game (N, U^1) are: $U^1(\{1\}) = U^1(\{1, 2\}) = U^1(\{1, 3\}) = U^1(N) = 8$, $U^1(\{2\}) = 3$, $U^1(\{3\}) = U^1(\{2, 3\}) = 5$. Accordingly, the convex hull of the following different points (8, 0, 0), (5, 3, 0), (3, 3, 2), (3, 0, 5), is the core of (N, U^1), which is a subset of the core of (N, C).

By increasing p_1 to 100 and leaving the other parameters intact, we get a different game (N, C') whose characteristic function values are: $C'(\{1\}) = 25$, $C'(\{2\}) = C'(\{3\}) = C'(\{2, 3\}) = 5$, $C'(\{1, 2\}) = 13$, $C'(\{1, 3\}) = 10$, $C'(N) = 8.5$ According to Property 4, as $C'(N) = U^2(N)$, the core of (N, U^2) is a subset of the core of the game (N, C'). The characteristic function of the game (N, U^2) is defined by $U^2(\{1\}) = 25$, $U^2(\{2\}) = 5$, $U^2(\{3\}) = 4$, $U^2(\{1, 2\}) = 13$, $U^2(\{1, 3\}) = 10$, $U^2(\{2, 3\}) = 4.375$, $U^2(N) = 8.5$. (N, U^2) is a centralizing aggregation game that has the decreasing differences property. According to Theorem 6, the auxiliary game $(N, \widetilde{U^2})$ is concave and its core coincides with the nonnegative core of (N, U^2). The values of the characteristic function of the game $(N, \widetilde{U^2})$ are: $\widetilde{U^2}(\{2\}) = \widetilde{U^2}(\{2,3\}) = 4.375$, $\widetilde{U^2}(\{3\}) = 4$, $\widetilde{U^2}(\{1,2\}) = \widetilde{U^2}(\{1,3\}) = \widetilde{U^2}(N) = \widetilde{U^2}(\{1\}) = 8.5$. The core of the concave game $(N, \widetilde{U^2})$ is defined by the convex hull of the extreme points of the polytope, namely: (8.5, 0, 0), (4.125, 4.375, 0), (4.5, 0, 4), (4.125, 0.375, 4), which coincides with the nonnegative core of (N, U^2) and is a subset of the core the game (N, C').

As a side comment note that similarly to the proof that the game (N, U^2) satisfies the conditions required by Theorem 6, it is possible to show that also the non-negative weighted mean game (N, U), where each element $\alpha_i > 0$, $1 \le i \le n$, is associated with a weight $w_i > 0$, and the characteristic function is $U(S) = \frac{\sum_{i \in S} w_i \alpha_i}{\sum_{i \in S} w_i}$ for any $S \subseteq N$, satisfies the same conditions, and therefore is totally balanced.

6.3 | The Common Cycle policy for the ELSP

The multiitem ELSP, introduced in Rogers (1958), considers the deterministic continuous time problem of producing a set $N = \{1, \ldots, n\}$ of products on a single machine. Each product $i \in N$ is associated with its constant demand rate d_i, its production rate p_i, its setup cost k_i for initiating a production run, and its inventory holding cost rate of h_i^{prime}. The machine can produce at most one product at a time, implying that there exists a feasible solution for the problem if and only if $\sum_{i=1}^{n} \rho_i < 1$, where $\rho_i = d_i / p_i$, for $i = 1, \ldots, n$, as otherwise, the machine's capacity is insufficient to meet the demands. The ELSP is the problem of minimizing the long-run average setup and inventory holding costs of scheduling the products on the machine so that the demands of all products are met on time, that is, neither stockouts nor backlogging are allowed. Setup times are assumed to be zero. More general variants of the ELSP have been considered in the literature, but, here, we focus on this simplest basic version. The ELSP is a natural extension of the simple Economic Order Quantity problem, but the restriction of producing at most one item at a time, makes the task of solving the ELSP a real challenge. Researchers have focused on finding optimal policies for the ELSP within given families of structured forms of policies. The family of policies that consists of the simplest form is called the family of *cyclic policies*, where each product is produced every T_i units of time. Finding the optimal policy within this class is equivalent to solving the following problem

$$\min\left\{\sum_{i=1}^{n}\frac{k_i}{T_i}+h_iT_i : \text{there exists a feasible cyclic policy } (T_i)_{i=1}^{n}\right\} \quad (2)$$

where $h_i = 0.5h_i' d_i(1-\rho_i)$. But even this restricted version with arbitrary cyclic schedules, has been proved in Gallego and Shaw (1997) to be NP-hard. If the problem is further restricted to a cyclic policy where all the products share the same cycle time, then the problem becomes easy. The optimal policy of this type, introduced in Hanssmann (1962) is called the *Common Cycle (CC)* policy: According to the CC policy, the products of any coalition $S \subseteq N$ share the same cycle length so that the long-run average total of its members is minimized. Next, we show that the cooperative game whose characteristic function returns the long-run average cost of the CC policy for any coalition $S \subseteq N$, is not balanced as it is a super-additive cost game. For this sake let, for any coalition $S \subseteq N$, $K(S) = \sum_{i\in S}k_i$ and $H(S) = \sum_{i\in S}h_i$. The optimal common cycle time of coalition $S \subseteq N$ is given by $T^{CC}(S) = \sqrt{\frac{K(S)}{H(S)}}$ implying that the cost of the coalition is $C^{CC}(S) = \sqrt{K(S)H(S)}$.

Claim 2 $C^{CC}(S) \geq C^{CC}(S_1) + C^{CC}(S_2)$ for any coalition $S \subseteq N$, and any two disjoint coalitions S_1 and S_2 that satisfy $S = S_1 \cup S_2$.

Proof The proof is immediate as in $C^{CC}(S)$ all products in the union $S = S_1 \cup S_2$ are required to have the same cycle time, where in $C^{CC}(S_1) + C^{CC}(S_2)$ the requirement is relaxed so that all products in S_1 need to have identical cycle times and all products in S_2 need to have identical cycle times. ■

In view of Claim 2, unless all ratios $\{k_i/h_i\}_{i=1,\ldots,n}$ are equal, the core of the ELSP game is empty. Researchers have proposed alternative techniques that generate cost allocations for unbalanced games as, for example: (a) the γ *core*—the efficiency constraint is relaxed, that is, the total cost allocated to all players is $\gamma C^{CC}(N)$ for $0 < \gamma < 1$, where γ is as large as possible (see Faigle & Kern, 1993); (b) the *least core*—where all the stand-alone constraints are relaxed, so that each coalition $S \subsetneq N$ is allocated a cost that is bounded from above by the cost of the coalition plus a constant $z > 0$, where z is as small as possible (see Maschler, Peleg & Shapley, 1979). Such directions are beyond the scope of this paper. Below we associate an alternative game to the CC policy, called *the CC setup frequency game,* a game that satisfies the sufficient condition proposed in this paper.

Suppose that in addition to the products' dependent setup and holding costs, the production facility incurs a fixed cost K_0 at each initialization of a common cycle. The fixed cost K_0 covers costs involved in the setup procedures carried out before the initialization of a common cycle. For accounting purposes, the management asks to find a stable cost allocation of $K_0/T^{CC}(N)$ among the products that reflects the effect of each product on the frequency of the cycle setups. This goal is achieved by investigating the core of the cooperative game (N, C^f), where $C^f(S) = \frac{K_0}{T^{CC}(S)} = \frac{K_0\sqrt{H(S)}}{\sqrt{K(S)}}$ for any $S \subseteq N$. For simplicity, let $K_0 = 1$.

Claim 3 The CC setup frequency game is totally balanced and its nonnegative core is fully identified.

Proof The characteristic function of the game (N, C^f) is of the form $C^f(S) = \sqrt{H(S)/K(S)}$ for any $S \subseteq N$. Recall from Section 1.1 that a set function of the form $H(S)/K(S)$ is a centralizing aggregation function. In view of Property 3, as the square root function is strictly increasing and concave, the conditions of Theorem 6 are satisfied, implying that the auxiliary game $(N, \widetilde{C^f})$ is concave. According to Theorem 2, the nonnegative core of (N, C^f) coincides with the core of its auxiliary game $(N, \widetilde{C^f})$, whose exact form is known. ■

The game (N, C^f) is not concave, as the following example shows: consider an ELSP instance with $n = 3$ products, where $h_1 = 9$, $k_1 = 1$, $h_2 = k_2 = 5$, $h_3 = 1$, and $k_3 = 9$. Thus, $C^f(\{1\}) = \sqrt{9/1} = 3$,. $C^f(\{1,2\}) = \sqrt{14/6} = 1.528$, $C^f(\{1,3\}) = \sqrt{10/10} = 1$, and $C^f(N) = \sqrt{15/15} = 1$. In order to rebut Property 1 on this example, we use $S = \{1\}$, $T = \{1, 2\}$ and $\ell = 3$, and get that $C^f(\{N\}) - C^f(\{1, 2\}) = 1–1.528 = -0.528 > C^f(\{1, 3\}) - C^f(\{1\}) = 1–3 = -2$.

7 | CONCLUSIONS

The determination of whether a given cooperative game has a nonempty core, let alone identifying core cost allocations, is a challenging task because of the exponential size of the problem. Thus, the identification of general sufficient conditions for proving the total balancedness of games may greatly simplify this task. In this article we add a new sufficient condition to this body of research: a regular centralizing aggregation game that has the decreasing variation property is totally balanced and, its nonnegative part of the core is fully characterized. Applications in queueing and scheduling games are presented.

An interesting open question deals with cooperative games where each of the players is associated with a positive score and the characteristic function is one of the various generalized mean functions. More specifically, let $(\alpha_1, \ldots, \alpha_n)$ be a vector of positive scores, and let p be a real number. The generalized mean that is associated with

$p \neq 0$ is $M_p(\alpha_1, \ldots, \alpha_n) = \left(\frac{1}{n}\sum_{i=1}^{n}\alpha_i^p\right)^{\frac{1}{p}}$, where M_1 is the arithmetic mean, $M_{-\infty}(\alpha_1, \ldots, \alpha_n) = \min\{\alpha_1, \ldots, \alpha_n\}$ and $M_{\infty}(\alpha_1, \ldots, \alpha_n) = \max\{\alpha_1, \ldots, \alpha_n\}$.

M_0 is defined as the geometric mean, that is, $M_0(\alpha_1, \ldots, \alpha_n) = (\Pi_{i=1}^{n}\alpha_i)^{\frac{1}{n}}$. It is easy to show that a game with a characteristic function that is the minimum score of a coalition, may have an empty core. The game whose characteristic function is the maximum score is the concave airport game discussed in Section 3, which is totally balanced and its core is fully characterized. The nonnegative arithmetic mean game satisfies the new sufficient condition, and therefore it is totally balanced and its nonnegative core is fully characterized.

A generalized mean for $-\infty < p < \infty$ is a special case of a weighted generalized mean, where the scores may have nonequal weights. Suppose that player i is associated with a score α_i and a weight $w_i = w_i(N) > 0$, such that $\sum_{i \in N} w_i = 1$. A weighted generalized mean for the scores in N is $M_p((\alpha_1 w_1), \ldots, (\alpha_n w_n)) = \left(\sum_{i=1}^{n} w_i \alpha_i^p\right)^{1/p}$ for $p \neq 0$, and $M_0((\alpha_1 w_1), \ldots, (\alpha_n w_n)) = \Pi_{i=1}^{n}\alpha_i^{w_i}$. A game whose characteristic function for $S \subseteq N$ is a weighted generalized mean of its scores, is defined similarly by using the weights $w_i(S)$ for $i \in S$, while calculating the cost of S, where $w_i(S) = \frac{w_i}{\sum_{j \in S} w_j}$, implying that $\sum_{i \in S} w_i(S) = 1$. An interesting question is to identify the weighted generalized mean games that are totally balanced.

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