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The Cost Allocation Problem for the First Order Interaction Joint Replenishment Model

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We consider an infinite-horizon deterministic joint replenishment problem with first order interaction. Under this model, the setup transportation/reorder cost associated with a group of retailers placing an order at the same time equals some group-independent major setup cost plus retailer-dependent minor setup costs. In addition, each retailer is associated with a retailer-dependent holding-cost rate. The structure of optimal replenishment policies is not known, thus research has focused on optimal power-of-two (POT) policies. Following this convention, we consider the cost allocation problem of an optimal POT policy among the various retailers. For this sake, we define a characteristic function that assigns to any subset of retailers the average-time total cost of an optimal POT policy for replenishing the retailers in the subset, under the assumption that these are the only existing retailers. We show that the resulting transferable utility cooperative game with this characteristic function is concave. In particular, it is a totally balanced game, namely, this game and any of its subgames have nonempty core sets. Finally, we give an example for a core allocation and prove that there are infinitely many core allocations.

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1. Introduction

In the business world of today it becomes more prevalent to lease the transportation/production and storage activities as well as other related services of a supply chain to an external third-party logistics (3PL) service provider. The use of 3PL started in the 1980s, but has grown significantly in recent years. Today many businesses prefer to lease several of their activities by a long-term commitment to a single 3PL company (see, e.g., Leahy et al. 1995). However, it is well recognized today that mainly large firms such as Minnesota Mining & Manufacturing Co. (3M), Eastman Kodak, Dow Chemical, Time Warner, and Sears Roebuck are leasing large parts of their logistical activities to 3PL providers. Small companies tend to be more skeptical regarding the advantage of using 3PL (see, e.g., Simchi-Levi et al. 2000). One of the main reasons for the reservations of small companies about doing business through a 3PL provider is related to the cost schemes offered by the 3PL providers, which often contain some economies-of-scale benefits that cannot be exploited by small firms.

In the last two decades, the importance of joining forces within a supply chain to reduce the systemwide costs has been strongly recognized by the OR/MS research community, as well as by practitioners. As a result, the body of research on joint replenishment problems through a 3PL provider has flourished. The main emphasis of this research has been on searching for policies that minimize the total systemwide costs for various systems under different assumptions. For a comprehensive review of supply chain management, see Tayur et al. (1999) or Simchi-Levi et al. (2004). However, a further question must be asked once the policy minimizing the total cost is found, which is how to allocate the total cost among the various parties in the supply chain. The cost allocation problem is important for cost accounting purposes as well as for enabling the management to decide on the profitability of the various entities in the supply chain. The cost allocation scheme may have a significant impact on the long-term strategic decisions, and therefore must be fair in the sense that no facility would feel that it was subsidizing the others. Intuitively, the cost allocation scheme should have the property that each party would feel that acting as a coalition is worthwhile for its own sake. Sharkey (1995) provides an excellent review of cost allocation problems in the context of transportation models.

In the context of the economic order quantity (EOQ) inventory model with safety stock, Gerchak and Gupta (1991) were the first to deal with the issue of how various stores can benefit from consolidating orders. They also raise the issue of how to split the gains (which they prove to be positive) due to consolidation, and suggest some ad hoc approaches. Their model assumes stochastic but independent demands and lead times and identical cost parameters across stores.

Later Robinson (1993) and Hartman and Dror (1996) cast this problem in a cooperative game framework and considered various solution concepts such as the core and the Shapley value. Hartman and Dror (2000) and Muller et al. (2002) look into the newsvendor problem. In particular, they define a model that calls for centralized orders and for sharing the cooperation gains among individual players. The latter among these two papers shows that the core of the game is nonempty under basically no requirements on the joint distribution of stochastic demands. However, the cost parameters ought to be uniform across vendors.

In this research, we focus on the cost allocation problem in an infinite-horizon deterministic single-warehouse joint replenishment model, where a number of retailers, each facing a constant demand rate, lease the reordering or transportation of their supplies, as well as their storage activities, to a 3PL provider. We assume that each time a delivery is requested by any subset of retailers, a fixed reorder/transportation (leasing or renting) cost, called *major setup cost*, is charged. Moreover, each retailer is associated with his own retailer-dependent fixed reorder/transportation cost, called *minor setup cost*, which is possibly a function of the distance or the travel time between the warehouse and the retailer. This cost is being incurred whenever the retailer replenishes its stock. The warehouse's costs are assumed to be exogenous to the model. We note that the same setup cost structure may also occur in a production process of a number of different items, where setting up the process for production of any subset of the items incurs the major setup cost, in addition to item-dependent setup costs incurred whenever the items are produced. We assume here that the fixed costs are exogenous to the model, and are not the outcome of a price discount mechanism determined by the model. In addition to the setup costs, the retailers pay the 3PL provider for holding their stock at depots in the retailers' sites.

The optimization problem associated with the above model is that of when to place orders for the various retailers, and what are the quantities to order each time a replenishment takes place. The goal is to minimize the time-average systemwide costs. Had the major setup cost been zero, the problem would in fact be that of solving independent EOQ problems. In particular, each retailer would order its EOQ in equidistant time intervals when its stock level is zero. The case where the major setup cost is positive calls for coordination of the timing of various orders for the sake of placing joint orders, making the optimization problem more intriguing. The model considered here with joint setup cost (see Equation (1)) is known in the literature as the *first-order interaction model* (see, e.g., Federgruen and Zheng 1995).

More involved cases are also considered in the literature. For example, see Federgruen and Zheng (1992) and Federgruen et al. (1992). We would also like to mention Meca et al. (2004), which deals with the special case where the minor setup costs are zero for all retailers. Finally, Dror and Hartman (2005) consider the same model, but their definition for the characteristic function of the cooperative game is different than ours.

The first-order interaction model is the simplest model that involves cooperation among retailers. In spite of its relative simplicity, the structure of optimal policies for this problem is as yet unknown, except for the zero-inventory-ordering (ZIO) property, which insures that under any optimal replenishment policy, each retailer orders only when its inventory level is zero. Thus, practitioners resorted to suboptimal policies that are efficient in terms of the computational effort and which have some guaranteed (hopefully, small) deviation from the optimal average-time total cost. In particular, we refer here to *power-of-two* (*POT*) *policies*, in which each retailer orders in equidistant time intervals that is an integer (positive or negative) power-of-two times a fixed base time unit. The optimal POT policy is known (see Jackson et al. 1985) to yield an average cost that is at most 6% higher than the optimal average-time total cost. By optimizing over the base time unit, the worst-case gap can be reduced to 2% (see Roundy 1985). Our results regarding the cost allocation hold for any fixed value of the base time unit, and in particular, for the optimal one.

Given the optimal POT ordering policy, the next question is how the retailers should split the total costs among themselves. Many options exist here. For example, a naive allocation can be that all pay their holding costs, and whenever an order is placed, all ordering retailers pay their minor setup cost and evenly share the major setup cost incurred. This scheme has the advantage of being simple, aesthetic, and maybe easy to argue on nontheoretical grounds. However, it is possible that some retailers may feel that they pay more than others toward the common goal of minimizing social costs. It fact, they may end up subsidizing the others. A specific example of that is given later in the paper. Thus, a more systematic approach is needed. We later define a cooperative transferable utility game representing the above-posed allocation problem and suggest the application of a game-based cost-sharing rule, the core.

For any subset of retailers, we associate a cost value that is the average-time total ordering/transportation and holding costs for the above-posed problem when only the retailers in the subset are present and when the optimal POT policy (given this set) is utilized. Further, we show that these costs values induce a cooperative game with transferable utility. We next consider the question of how to allocate the total systemwide cost among the various retailers. In other words, we look for a cost-sharing (or Pareto efficient) solution concept. The main purpose of this paper is to show that the abovementioned game is concave, and to present an example for a core allocation. Recently, Dror and Hartman (2005) considered a similar problem where the definition of the value of a coalition is the cost under the optimal replenishment policy. They give an example where this cooperative game has an empty core. Finally, they state a sufficient condition for the nonemptiness for the core of this game.

The rest of this paper is organized as follows. The next section contains notations and preliminaries. Section 3 states our main results. In particular, we show there that the suggested game is concave. In §4, a core allocation is introduced, and the core is shown to contain an infinite number of allocations. Section 5 concludes the paper.

2. Preliminaries and Notation

We consider the cost allocation problem in an infinite-horizon single-warehouse joint replenishment model, with n retailers in the set $N = \{1, 2, \ldots, n\}$. The cost structure considered here is as follows: Each time a delivery is requested by any subset of retailers, a fixed reorder/transportation cost K_0, called the *major setup cost*, is charged. Moreover, each retailer i is associated with a retailer-dependent fixed reorder/transportation cost K_i, called the *minor setup cost*. Thus, if a set S, $\varnothing \subset S \subseteq N$, of retailers orders simultaneously, the setup cost incurred at that time is

$$K_0 + \sum_{i\in S} K_i. \tag{1}$$

Each retailer i, $1 \leqslant i \leqslant n$, is assumed to face a retailer-dependent deterministic, constant demand rate denoted by d_i, and the cost of holding one unit of product for one unit of time at this retailer is h_i. To simplify notation, we let $g_i = h_i d_i/2$ for $1 \leqslant i \leqslant n$ be the *holding-cost parameter* of retailer i. Finally, we assume zero lead times. Identical lead times can be handled similarly. Without loss of generality, we assume that the retailers in $N = \{1, \ldots, n\}$ are ordered such that $K_1/g_1 \leqslant K_2/g_2 \leqslant \cdots \leqslant K_n/g_n$. For convenience, we define a dummy retailer, retailer $n+1$, with $K_{n+1}/g_{n+1} = \infty$.

As mentioned in the introduction, we restrict ourselves to POT policies, where each retailer orders at equidistant time intervals of length $2^{m_i}B$ for some (positive or negative) integer m_i, $1 \leqslant i \leqslant n$, and for some common base time unit B. Jackson et al. (1985) showed that the optimal among such policies yields an average cost that is at most 6% higher than the optimal average-time cost. By optimizing over B, $B \in [1, 2)$, the worst-case gap can be reduced to 2% (see Roundy 1985). Our results regarding the cost allocation hold for any fixed value of $B \in [1, 2)$ and, in particular, for the optimal one. For the sake of simplicity, we assume in the sequel that time units are scaled so that $B = 1$.

For any retailer i, $1 \leqslant i \leqslant n+1$, let $\tau'_i = \sqrt{K_i/g_i}$ and let $\tau_i = \sqrt{(K_0 + K_i)/g_i}$. The sequence τ'_i is nondecreasing in i. Also, let T'_i and T_i be the POT rounding-off of τ'_i and τ_i, respectively.

Let $S = \{i_1, i_2, \ldots, i_s\} \subseteq N$ be a set of s retailers. Note that $s = |S|$, i.e., it is the cardinality of S. For S with $|S| \geqslant 1$, denote $K_0 + \sum_{i\in S} K_i$ by $K^0(S)$ and set $K^0(\varnothing) = 0$. Denote also $\sum_{i\in S} g_i$ by $G(S)$. Moreover, to simplify notation, let $i_0 = 0$ and $g_0 = 0$. Recall, however, that $K_0 > 0$. Also, let

$$i^*(S) = \arg\max\left\{k,\ 1 \leqslant k \leqslant s \,\middle|\, \frac{\sum_{0\leqslant j\leqslant k} K_{i_j}}{\sum_{0\leqslant j\leqslant k} g_{i_j}} \geqslant \frac{K_{i_k}}{g_{i_k}}\right\}. \tag{2}$$

Note that $i^*(S) \geqslant 1$. Let $S^0 = \{i_1, \ldots, i^*(S)\}$. From Jackson et al. (1985), we learn that the optimal POT policy for the retailers in S is as follows. The retailers in S^0 order simultaneously every 2^{m_0} time units, where 2^{m_0} is the integer POT closest to $\tau_{\min}(S)$, where

$$\tau_{\min}(S) = \sqrt{\frac{K^0(S^0)}{G(S^0)}}, \tag{3}$$

i.e., m_0 is the unique integer that satisfies the inequality $2^{m_0-0.5} \leqslant \tau_{\min}(S) < 2^{m_0+0.5}$. We denote the POT reorder interval of S^0, namely, 2^{m_0}, by $T_{\min}(S)$. If $i^*(S) < i_s$, then each of the retailers $i_j \in S\backslash S^0$ orders at most as frequently as the set S^0. Indeed, each time such a retailer orders, the set S^0 also orders (but not the other way around). In fact, a retailer i_j with $i_j \in S\backslash S^0$ orders at times prescribed by its individual EOQ model, i.e., at τ'_{i_j}, rounded to the closest integer POT, named T'_{i_j}. Therefore, we refer to S^0 as the *minimal set* of S because the retailers in S^0 replenish their stocks the most frequently among all retailers in S.

We use also the following notation: For a set $S \subseteq N$, let $i^{*+}(S) = \min\{n+1, \min\{i_j \mid i_j \in S\backslash S^0\}\}$. Clearly, if $i^*(S) < i_s$, then $T_{\min}(S) \leqslant T'_{i^{*+}(S)} \leqslant \cdots \leqslant T'_{i_s}$. The optimal average-time total ordering plus holding costs of the set of retailers S, when they restrict themselves to POT policies (namely, $v(S)$) equals the optimal objective value of the following integer program:

$$\text{(JRPPT}(S)) \quad \text{Min} \quad \sum_{j=0}^{s}\left(\frac{K_{i_j}}{t_{i_j}} + g_{i_j} t_{i_j}\right)$$
$$\text{s.t.} \quad t_{i_j} = 2^{m_{i_j}}, \quad 0 \leqslant j \leqslant s,$$
$$m_{i_j} \geqslant m_0, \quad 1 \leqslant j \leqslant s,$$
$$m_{i_j} \text{ integer}, \quad 0 \leqslant j \leqslant s.$$

(JRPPT(S)) was solved in Jackson et al. (1985) and was shown to have an optimal objective value of the form

$$v(S) = \frac{K^0(S^0)}{T_{\min}(S)} + T_{\min}(S) G(S^0) + \sum_{i_j \in S\backslash S^0}\left(\frac{K_{i_j}}{T'_{i_j}} + g_{i_j} T'_{i_j}\right).$$

Example 1. Consider the following example with two retailers. Let $K_0 = 15$, $K_1 = K_2 = 1$, $g_1 = 1$, and $g_2 = 1/64$. It is possible to see that $\tau_1 = 4$, $\tau'_2 = 8$, and $\tau_2 = 32$. Moreover, the minimal set is $S^0 = \{1\}$, i.e., it contains only Retailer 1. This retailer orders every four units of time, while Retailer 2 joins him every other order. The total average cost of this policy is 8.25. In the case of no cooperation, each of the retailers will order according to the EOQ formula, with a setup cost of 16 per order. As a result, Retailer 1 will maintain his replenishment policy of ordering every four time units at an average cost of eight, while Retailer 2 will replenish only every 32 time units at an average cost of one. (Thus, acting independently increases the total average cost by 0.75.) There is a feeling here that by using the POT joint replenishment policy, Retailer 2 contributes more toward the joint venture because by joining forces the major setup costs associated with his separate orders are in fact eliminated.

As $v(\varnothing)=0$ and as $v(S)$ is a real number, the pair (N,v) defines a cooperative game with transferable utility. Moreover, as it is clear that for any coalition S, $\varnothing\subseteq S\subseteq N$, $v(S)+v(N\backslash S)\geqslant v(N)$, the formation of the grand coalition is a natural outcome from a bargaining process. The next question is how the cost $v(N)$ has to be allocated among the retailers. In other words, we look for a cost-sharing (or Pareto efficient) solution concept. For this sake we refer to the following definitions.

A game (N,v) is said to be *concave* if the following property holds:

$$v(R\cup\{l\})-v(R)\geqslant v(S\cup\{l\})-v(S) \quad \text{for any } R\subset S\subset N,\ l\in N\backslash S.$$

Also, a vector $x\in R^n$ is said to be a *core allocation* for the game (N,v) if $\sum_{i\in N}x_i=v(N)$ and if for any set of retailers S with $\varnothing\subseteq S\subseteq N$, $\sum_{i\in S}x_i\leqslant v(S)$. A game is called *balanced* if its core is not empty, and it is called *totally balanced* if all the games with the same characteristic function but restricted to subsets of players, are balanced too. It is well known that a concave game is totally balanced.

In the next section, we state our main result.

3. The Concave Inventory Game

In this section, we prove that the first-order interaction joint replenishment model described above induces a concave cooperative game. As a result, a fair cost allocation exists, such that no subset S of retailers, $\varnothing\subset S\subseteq N$, would have an incentive to deviate from the grand coalition to reduce its total costs. We first present our main theorem:

THEOREM 1. *The transferable utility cooperative game with N as its set of players and with $v(S)$, $\varnothing\subseteq S\subseteq N$, as its characteristic function, is concave. In particular, it is totally balanced.*

We defer the proof of the theorem to the end of this section and start by proving some properties relating the optimal POT policies for two subsets of retailers R and S with $R\subset S\subseteq N$. The proof of the theorem makes use of the following definitions and lemmas:

LEMMA 1. *For any subset of retailers $C\subseteq S^0$ and any retailer $i_j\in S^0$,*

$$\frac{K^0(C)}{G(C)}\geqslant\frac{K_{i_j}}{g_{i_j}}.$$

PROOF. See Jackson et al. (1985). □

LEMMA 2. *For any j with $i_j\in S$, $i_{j+1}\in S$, and $i_j<i^*(S)$,*

$$\frac{K^0(\{i_1,\ldots,i_j\})}{G(\{i_1,\ldots,i_j\})}\geqslant\frac{K^0(\{i_1,\ldots,i_{j+1}\})}{G(\{i_1,\ldots,i_{j+1}\})}.$$

PROOF. Aiming for a contradiction, let $i_j<i^*(S)$ be such that the lemma's inequality does not hold. This implies that for this i_j,

$$\frac{K_{i_{j+1}}}{g_{i_{j+1}}}>\frac{K^0(\{i_1,\ldots,i_j\})}{G(\{i_1,\ldots,i_j\})}.$$

This contradicts Lemma 1. □

LEMMA 3.
1. $\tau'_{i^*(S)}\leqslant\tau_{\min}(S)\leqslant\tau_{i_1}$.
2. $\tau_{\min}(S)<\tau'_{i^{*+}(S)}$.

PROOF. 1. If $i^*(S)=i_1$, then $\tau_{\min}(S)=\tau_{i_1}\geqslant\tau'_{i_1}$. Otherwise, Lemma 2 implies that $\tau_{\min}(S)\leqslant\tau_{i_1}$, and Lemma 1 implies that $\tau_{\min}(S)\geqslant\tau'_{i^*(S)}$.

2. If $S^0=S$, then $i^{*+}(S)=n+1$ and the inequality holds. Otherwise, by the definition of $i^*(S)$,

$$\frac{K^0(S^0\cup\{i^{*+}(S)\})}{G(S^0\cup\{i^{*+}(S)\})}<\frac{K_{i^{*+}(S)}}{g_{i^{*+}(S)}}.$$

Thus,

$$\tau_{\min}(S)=\sqrt{\frac{K^0(S^0)}{G(S^0)}}<\sqrt{\frac{K^0(S^0\cup\{i^{*+}(S)\})}{G(S^0\cup\{i^{*+}(S)\})}}<\sqrt{\frac{K_{i^{*+}(S)}}{g_{i^{*+}(S)}}}=\tau'_{i^{*+}(S)}.\quad\square$$

LEMMA 4. *For any $S\subseteq N$, $i^{*+}(S)>i^*(N)$.*

PROOF. If $i^{*+}(S)=n+1$, then the claim is trivial. Assume now that $i^{*+}(S)\leqslant n$. Recall that $i^*(N)$ is the largest index $i\in N$ that satisfies the inequality

$$\frac{K_i}{g_i}\leqslant\frac{K^0(\{1,\ldots,i\})}{G(\{1,\ldots,i\})}$$

and that $i^*(S)$ is the largest index $i\in S$ that satisfies the inequality

$$\frac{K_i}{g_i}\leqslant\frac{K^0(S\cap\{1,\ldots,i\})}{G(S\cap\{1,\ldots,i\})}.$$

Note that by Lemma 1 the last inequality holds for any i, $i\leqslant i^*(N)$. Thus, any $i\in N^0\cap S$ is also in S^0. This completes the proof. □

LEMMA 5. *For $S\subseteq N$, $\tau_{\min}(N)\leqslant\tau_{\min}(S)$.*

Proof. To prove the lemma, we first prove the following inequality:

$$\frac{K^0(\{1,\ldots,l\})}{G(\{1,\ldots,l\})} \leqslant \frac{K^0(\{1,\ldots,l\}\cap S)}{G(\{1,\ldots,l\}\cap S)}, \quad l\in S. \tag{4}$$

To prove this inequality, we denote the indices of the retailers in S by $i_1 < i_2 < \cdots < i_k = l$. We also let the subset of indices L_p, for $p = 1,\ldots,k$, be defined as follows: $L_p = \{i\in N\colon i_{p-1} < i \leqslant i_p\}$. Therefore, L_p is a consecutive subset of N, where only its largest indexed member, namely i_p, is also in S. Clearly,

$$\frac{\sum_{i\in L_p} K_i}{\sum_{i\in L_p} g_i} \leqslant \frac{K_{i_p}}{g_{i_p}}.$$

Now,

$$\frac{K^0(\{1,\ldots,l\})}{G(\{1,\ldots,l\})} = \frac{K_0+\sum_{p=1}^k\sum_{i\in L_p}K_i}{\sum_{p=1}^k\sum_{i\in L_p}g_i} \leqslant \frac{K_0+\sum_{p=1}^k K_{i_p}}{\sum_{p=1}^k g_{i_p}} = \frac{K^0(\{1,\ldots,l\}\cap S)}{G(\{1,\ldots,l\}\cap S)},$$

which completes the proof of inequality (4).

To conclude the proof of the lemma, we distinguish between two cases. In the first case, $i^*(S)\leqslant i^*(N)$. Here, the proof follows by substituting $i^*(S)$ for l in (4) and using Lemma 2 for showing that

$$\frac{K^0(S^0)}{G^0(S^0)} \geqslant \frac{K^0(\{1,\ldots,i^*(S)\})}{G(\{1,\ldots,i^*(S)\})} \geqslant \frac{K^0(\{1,\ldots,i^*(N)\})}{G(\{1,\ldots,i^*(N)\})}.$$

In the second case, $i^*(S) > i^*(N)$. Now, by Lemma 1,

$$\frac{K_{i^*(S)}}{g_{i^*(S)}} \leqslant \frac{K^0(S^0)}{G(S^0)}.$$

Thus, invoking Part 2 of Lemma 3,

$$\tau^2_{\min}(N) = \frac{K^0(N^0)}{G(N^0)} < \frac{K_{i^*(N)+1}}{g_{i^*(N)+1}} \leqslant \frac{K_{i^*(S)}}{g_{i^*(S)}} \leqslant \frac{K^0(S^0)}{G(S^0)} = \tau^2_{\min}(S). \quad \square$$

We further need the following notation. We will denote the set $S\cup\{l\}$ for $l\in N\backslash S$ by S_l. Note that the minimal set of S_l, i.e., S_l^0, can coincide with the minimal set of S, namely, with S^0, or it can contain l. It is also possible that $\{l\}$ is the minimal set of S_l. Lemma 4 implies that $i^*(S_l) < i^{*+}(S)$.

Proof of Theorem 1. Recall that the sets of retailers R and S are such that $R\subset S\subset N$ and we need to show that for any retailer $l\in N\backslash S$, $v(R_l)-v(R)\geqslant v(S_l)-v(S)$. Toward this end, we write the value of various coalitions explicitly. As the optimal replenishment interval for retailer i is a function of the underlying set on which the algorithm is applied, we denote by $T_i^{\text{opt}}(S)$ the replenishment interval for retailer i when the algorithm is applied on set S. As follows from Lemma 4, $i^*(S_l) < i^{*+}(S)$; hence, it is clear that for all $i\in S$ with $i > i^*(S)$, $T_i^{\text{opt}}(S) = T_i^{\text{opt}}(S_l) = T_i'$, which is the POT rounding-off of τ_i'. Therefore,

$$\begin{aligned} v(S)-v(S_l) &= \frac{K^0(S^0)}{T_{\min}(S)} + T_{\min}(S)G(S^0) - \left(\frac{K_l}{T_l^{\text{opt}}(S_l)} + g_lT_l^{\text{opt}}(S_l)\right) \\ &\quad - \left(\frac{K^0(S_l^0\backslash\{l\})}{T_{\min}(S_l)} + T_{\min}(S_l)G(S_l^0\backslash\{l\})\right) \\ &\quad - \sum_{i=i^{*+}(S_l),\, i\in S}^{i^*(S)} \left(\frac{K_i}{T_i'} + g_iT_i'\right) \end{aligned}$$

and

$$\begin{aligned} v(R)-v(R_l) &= \frac{K^0(R^0)}{T_{\min}(R)} + T_{\min}(R)G(R^0) - \left(\frac{K_l}{T_l^{\text{opt}}(R_l)} + g_lT_l^{\text{opt}}(R_l)\right) \\ &\quad - \left(\frac{K^0(R_l^0\backslash\{l\})}{T_{\min}(R_l)} + T_{\min}(R_l)G(R_l^0\backslash\{l\})\right) \\ &\quad - \sum_{i=i^{*+}(R_l),\, i\in R}^{i^*(R)} \left(\frac{K_i}{T_i'} + g_iT_i'\right). \end{aligned}$$

We first show that $T_l' \leqslant T_l^{\text{opt}}(S_l) \leqslant T_l^{\text{opt}}(R_l)$. If l is not a member of either the minimal set of S_l or of the minimal set of R_l, then $T_l^{\text{opt}}(S_l) = T_l^{\text{opt}}(R_l) = T_l'$. If l is in the minimal set of S_l, then, by Lemma 4, it is also a member of the minimal set of R_l. If l is a member of both minimal sets of S_l and R_l, then $T_l^{\text{opt}}(S_l) = T_{\min}(S_l)$ and $T_l^{\text{opt}}(R_l) = T_{\min}(R_l)$. By invoking Lemma 5, $T_l^{\text{opt}}(S_l) \leqslant T_l^{\text{opt}}(R_l)$ and by Part 1 of Lemma 3, $T_l^{\text{opt}}(S_l) \geqslant T_l'$. Otherwise, l is a member of the minimal set of R_l, but is not a member of the minimal set of S_l. Thus, in S_l, $T_l^{\text{opt}}(S_l) = T_l'$, where in R_l, $T_l^{\text{opt}}(R_l) = T_{\min}(R_l)$. By Part 1 of Lemma 3, $\tau_{\min}(R_l) \geqslant \tau_l'$, and therefore also $T_{\min}(R_l) \geqslant T_l'$, proving that $T_l' \leqslant T_l^{\text{opt}}(S_l) \leqslant T_l^{\text{opt}}(R_l)$. Because T_l' is the minimizer of $\{(K_l/x)+g_lx\colon x = 2^{m_l}$ and m_l is an integer$\}$, we obtain that

$$\frac{K_l}{T_l^{\text{opt}}(S_l)} + g_lT_l^{\text{opt}}(S_l) \leqslant \frac{K_l}{T_l^{\text{opt}}(R_l)} + g_lT_l^{\text{opt}}(R_l).$$

Therefore, it is sufficient to show that

$$\begin{aligned} &\frac{K^0(S^0)}{T_{\min}(S)} + T_{\min}(S)G(S^0) - \left(\frac{K^0(S_l^0\backslash\{l\})}{T_{\min}(S_l)} + T_{\min}(S_l)G(S_l^0\backslash\{l\})\right) - \sum_{i=i^{*+}(S_l),\, i\in S}^{i^*(S)} \left(\frac{K_i}{T_i'} + g_iT_i'\right) \\ &\geqslant \frac{K^0(R^0)}{T_{\min}(R)} + T_{\min}(R)G(R^0) - \left(\frac{K^0(R_l^0\backslash\{l\})}{T_{\min}(R_l)} + T_{\min}(R_l)G(R_l^0\backslash\{l\})\right) - \sum_{i=i^{*+}(R_l),\, i\in R}^{i^*(R)} \left(\frac{K_i}{T_i'} + g_iT_i'\right). \end{aligned} \tag{5}$$

To prove inequality (5), we distinguish between two cases:

Case 1. $i^*(R) \leqslant i^*(S)$. We will show that the following two inequalities hold. The first is

$$\frac{K^0(S^0)}{T_{\min}(S)} + T_{\min}(S)G(S^0) \geqslant \frac{K^0(R^0)}{T_{\min}(R)} + T_{\min}(R)G(R^0) + \frac{\sum_{i=1,\, i\in S\setminus R}^{i^*(S_l)} K_i}{T_{\min}(S_l)} + T_{\min}(S_l)\sum_{i=1,\, i\in S\setminus R}^{i^*(S_l)} g_i + \sum_{i=i^{*+}(S_l),\, i\in S\setminus R}^{i^*(S)} \left(\frac{K_i}{T_i'} + g_i T_i'\right)$$

and the second is

$$\frac{K^0(R_l^0\setminus\{l\})}{T_{\min}(R_l)} + T_{\min}(R_l)G(R_l^0\setminus\{l\}) + \sum_{i=i^{*+}(R_l),\, i\in R}^{i^*(R)} \left(\frac{K_i}{T_i'} + g_i T_i'\right) \geqslant \frac{K_0 + \sum_{i=1,\, i\in R}^{i^*(S_l)} K_i}{T_{\min}(S_l)} + T_{\min}(S_l)\sum_{i=1, i\in R}^{i^*(S_l)} g_i + \sum_{i=i^{*+}(S_l),\, i\in R}^{i^*(S)} \left(\frac{K_i}{T_i'} + g_i T_i'\right).$$

These two inequalities imply inequality (5). To prove the first among these two inequalities, we note that its left-hand side equals $v(S^0)$, i.e., the optimal POT policy average-time cost of the minimal set of S. This is due to the fact that when the algorithm is applied to a subset that is the minimal set of some other set, it will end up with a new minimal set that is the subset itself, i.e., $(S^0)^0 = S^0$. Lemma 4 implies that $i^{*+}(R) > i^*(S)$, and because $i^*(R) \leqslant i^*(S)$ in this case, we get that $R^0 \cup (S^0\setminus R) = S^0$. Thus, the right-hand side of the first inequality is a solution to the (JRPPT(S^0)) problem, except that now some of the constraints $m_i \geqslant m_0$ are relaxed: The constraints $m_i \geqslant m_0$ for $i \in R^0$ stay intact, while for $i \in S^0\setminus R$ they are replaced by the constraints $m_i \geqslant \log_2(T_{\min}(S_l))$. By Lemma 5, $T_{\min}(S_l) \leqslant T_{\min}(S)$, implying that each such constraint is relaxed. Thus, the solution to the relaxed version of the (JRPPT(S^0)) problem is a lower bound on $v(S^0)$.

As for the second inequality, we note by Lemma 4 that the two sets $S^0\setminus R^0$ and R are disjoint. Thus, in the last term of the right-hand side of the second inequality, one can replace the upper summation limit $i^*(S)$ by $i^*(R)$, which results in the following equivalent inequality:

$$\frac{K^0(R_l^0\setminus\{l\})}{T_{\min}(R_l)} + T_{\min}(R_l)G(R_l^0\setminus\{l\}) + \sum_{i=i^{*+}(R_l),\, i\in R}^{i^*(R)} \left(\frac{K_i}{T_i'} + g_i T_i'\right) \geqslant \frac{K_0 + \sum_{i=1,\, i\in R}^{i^*(S_l)} K_i}{T_{\min}(S_l)} + T_{\min}(S_l)\sum_{i=1,\, i\in R}^{i^*(S_l)} g_i + \sum_{i=i^{*+}(S_l),\, i\in R}^{i^*(R)} \left(\frac{K_i}{T_i'} + g_i T_i'\right).$$

In view of the fact that $i^*(R_l) \leqslant i^{*+}(R)$ (see Lemma 4), the left-hand side of the above inequality represents the optimal objective function value of the (JRPPT($R^0 \cup \{l\}$)) problem, excluding the average-time minor setup cost plus holding cost of retailer l. The resulting minimal set is replenished every $T_{\min}(R_l)$ units of time. The right-hand side represents the objective function value of a relaxation of (JRPPT($R^0 \cup \{l\}$)) (again excluding the average-time minor setup cost plus holding cost of retailer l). In the relaxation, each of the constraints $m_i \geqslant m_0$ for $i \in R^0$ is replaced by $m_i \geqslant \log_2(T_{\min}(S_l))$. By Lemma 5, $T_{\min}(S_l) \leqslant T_{\min}(R_l)$, therefore implying that each such constraint is relaxed. Thus, the solution to this relaxed version of the (JRPPT($R^0 \cup \{l\}$)) problem is a lower bound on the left-hand side value. This completes the proof for Case 1 of inequality (5).

Case 2. $i^*(R) > i^*(S)$. We prove the following two inequalities, which combined imply (5):

$$\frac{K^0(S^0)}{T_{\min}(S)} + T_{\min}(S)G(S^0) + \sum_{i=i^{*+}(S),\, i\in R}^{i^*(R)} \left(\frac{K_i}{T_i'} + g_i T_i'\right) \geqslant \frac{K^0(R^0)}{T_{\min}(R)} + T_{\min}(R)G(R^0) + \frac{\sum_{i=1,\, i\in S\setminus R}^{i^*(S_l)} K_i}{T_{\min}(S_l)} + \sum_{i=1, i\in S\setminus R}^{i^*(S_l)} g_i T_{\min}(S_l) + \sum_{i=i^{*+}(S_l),\, i\in S\setminus R}^{i^*(S)} \left(\frac{K_i}{T_i'} + g_i T_i'\right)$$

and

$$\frac{K^0(R_l^0\setminus\{l\})}{T_{\min}(R_l)} + T_{\min}(R_l)G(R_l^0\setminus\{l\}) + \sum_{i=i^{*+}(R_l),\, i\in R}^{i^*(R)} \left(\frac{K_i}{T_i'} + g_i T_i'\right) \geqslant \frac{K_0 + \sum_{i=1,\, i\in R}^{i^*(S_l)} K_i}{T_{\min}(S_l)} + \sum_{i=1,\, i\in R}^{i^*(S_l)} g_i T_{\min}(S_l) + \sum_{i=i^{*+}(S_l),\, i\in R}^{i^*(R)} \left(\frac{K_i}{T_i'} + g_i T_i'\right).$$

We start with proving the first inequality. The left-hand side of this inequality is the optimal objective value of the (JRPPT($S^0 \cup R^0$)) problem. The resulting minimal set is S^0 that is replenished every $T_{\min}(S)$ units of time. The right-hand side of the inequality represents a relaxation of the same problem, where each of the constraints $m_i \geqslant m_0$ for $i \in S^0\setminus R^0$ is replaced by the constraint $m_i \geqslant \log_2(T_{\min}(S_l))$. In view of Lemma 5, $T_{\min}(S_l) \leqslant T_{\min}(S)$, and thus we get that these constraints are a relaxation of the original ones. This completes the proof of the first inequality.

In proving the second inequality, note that its left-hand side corresponds to the optimal objective function value of (JRPPT($R^0 \cup \{l\}$)), excluding the average-time minor setup cost plus the holding cost of retailer l. Because $i^*(R_l) \leqslant i^{*+}(R)$ (see Lemma 4), the resulting minimal set is R_l^0, which is replenished every $T_{\min}(R_l)$ units of time. The right-hand side is the objective function value of (JRPPT($R^0 \cup \{l\}$)) (again excluding the average-time cost of retailer l), but where all the constraints $m_i \geqslant m_0$ are replaced by $m_i \geqslant \log_2(T_{\min}(S_l))$. In view of Lemma 5, $T_{\min}(S_l) \leqslant T_{\min}(R_l)$, and therefore the right-hand side is a relaxation of the left-hand side. This completes the proof of inequality (5) and, in fact, of the theorem. □

Note that $T_{\min}(N)$ is the time length between two orders, in each of which at least the retailers in N^0 replenish. At some of these replenishment points other retailers order too. As retailer n is the one who orders with the least frequency, its reorder points can be considered to be the beginning of a new inventory cycle. In particular, these are the only points in time in which all stocks are empty. In the next section, we derive a core allocation.

4. A Core Allocation

We next describe a core allocation for the game (N, v). We find this cost allocation appealing in terms of suggesting a fair way to split the total costs. Establishing that the suggested allocation is indeed a core allocation can be considered as an alternative proof for the balancedness of the game. However, recall that Theorem 1 says more than just balancedness. At the end of the section, we show that in fact there are infinitely many core allocations.

THEOREM 2. *There exists a core allocation under which all retailers pay their own minor setup costs and holding costs, and each retailer in N^0, the minimal set of N, pays part of the major setup cost. A retailer j, $j \notin N^0$, does not pay anything toward the major setup cost. In particular, if all the retailers in N^0 have the same cost characteristics—that is, $K_j = K$ and $h_j = h$—then the allocation of the major setup cost is nonincreasing in $j \in N$, and is strictly increasing in the demand rate of the retailers in N^0.*

PROOF. Let $\tau^2 = \tau^2_{\min}(N) = K^0(N^0)/G(N^0)$ and define θ_j as follows:

$$\theta_j = \begin{cases} \dfrac{g_j\tau^2 - K_j}{K_0}, & j \in N^0, \\[2ex] 0, & j \notin N^0. \end{cases}$$

We propose the following cost allocation x_j for $j \in N$:

$$x_j = \begin{cases} \dfrac{\theta_j K_0 + K_j}{T_{\min}(N)} + g_j T_{\min}(N), & j \in N^0, \\[2ex] \dfrac{K_j}{T'_j} + g_j T'_j, & j \notin N^0. \end{cases}$$

We show below that $x \in R^n$ is a core allocation for the game (N, v).

It is easy to verify that $\sum_{j=1}^n \theta_j = 1$. It is also easy to see that the allocation is efficient, i.e., $\sum_{j=1}^n x_j = v(N)$. In the special case that $K_j = K$ and $h_j = h$ for all $j \in N^0$,

$$\theta_j = \begin{cases} \dfrac{(hd_j/2)\tau^2 - K}{K_0}, & j \in N^0, \\[2ex] 0 & j \notin N^0, \end{cases}$$

which is increasing in the demand rate of the retailers within the minimal set. We next show that for all $S \subseteq N$,

$$\sum_{j\in S} x_j \leqslant v(S). \tag{6}$$

Recall that

$$\begin{aligned} v(S) &= \frac{K^0(S^0)}{T_{\min}(S)} + T_{\min}(S)G(S^0) + \sum_{j=i^{*+}(S),\, j\in S}^{n} \left(\frac{K_j}{T'_j} + g_j T'_j\right) \\ &= \frac{K^0(S^0\cap N^0)}{T_{\min}(S)} + G(S^0\cap N^0)T_{\min}(S) \\ &\quad + \sum_{j=\min\{i^*(S),\, i^*(N)\}+1,\, j\in S}^{\max\{i^*(S),\, i^*(N)\}} \left(\frac{K_j}{T_{\min}(S)} + g_j T_{\min}(S)\right) \\ &\quad + \sum_{j=i^{*+}(S),\, j\in S}^{n} \left(\frac{K_j}{T'_j} + g_j T'_j\right), \end{aligned}$$

where the last equation follows from Lemma 4. We first note that any retailer $j \in S$, $j > \max\{i^*(S), i^*(N)\}$, satisfies $j \geqslant i^{*+}(S)$, and it pays, under the suggested allocation, the amount of $x_j = (K_i/T'_j) + g_j T'_j$. Thus, proving inequality (6) is equivalent to proving the following inequality:

$$\begin{aligned} &\frac{K^0(S^0\cap N^0)}{T_{\min}(S)} + G(S^0\cap N^0)T_{\min}(S) \\ &\quad + \sum_{j=\min\{i^*(S),\, i^*(N)\}+1,\, j\in S}^{\max\{i^*(S),\, i^*(N)\}} \left(\frac{K_j}{T_{\min}(S)} + g_j T_{\min}(S)\right) \\ &\qquad \geqslant \sum_{j=1,\, j\in S}^{\max\{i^*(S),\, i^*(N)\}} x_j. \end{aligned} \tag{7}$$

We distinguish between two cases: (1) $i^*(S) \leqslant i^*(N)$, and (2) $i^*(S) > i^*(N)$. We will show that in both cases it is sufficient to prove the following inequality:

$$\frac{K^0(S^0\cap N^0)}{T_{\min}(S)} + G(S^0\cap N^0)T_{\min}(S) \geqslant \sum_{j\in S^0\cap N^0} x_j. \tag{8}$$

Case 1. From Lemma 4, it follows that if $i^*(S) \leqslant i^*(N)$, then the sets S and $\{i^*(S)+1, \ldots, i^*(N)\}$ are disjoint. In such a case, the summation in the left-hand side of inequality (7) is over an empty set, and indeed it remains to prove only (8).

Case 2. If $i^*(S) > i^*(N)$, then for any $i \in \{i^*(N)+1, \ldots, i^*(S)\} \cap S$,

$$x_i = \frac{K_i}{T'_i} + g_i T'_i \leqslant \frac{K_i}{T_{\min}(S)} + g_i T_{\min}(S),$$

as T'_i is the POT minimizer of the EOQ cost function $f(t) = (K_i/t) + g_i t$. Thus, in this case too, in order to prove inequality (7), it is sufficient to prove (8).

We proceed now by proving (8). Note that for any retailer $i \in S^0 \cap N^0$,

$$x_i = \frac{\theta_i K_0 + K_i}{T_{\min}(N)} + g_i T_{\min}(N).$$

Thus, proving (8) is equivalent to proving that

$$\frac{K^0(S^0 \cap N^0)}{T_{\min}(S)} + G(S^0 \cap N^0) T_{\min}(S) \geqslant \frac{\sum_{i \in S^0 \cap N^0} (\theta_i K_0 + K_i)}{T_{\min}(N)} + G(S^0 \cap N^0) T_{\min}(N).$$

To prove this last inequality, note that (1) $\sum_{i \in S^0 \cap N^0} \theta_i \leqslant 1$ and therefore, $\sum_{i \in S^0 \cap N^0} (\theta_i K_0 + K_i) \leqslant K^0(S^0 \cap N^0)$; and that (2) $T_{\min}(N)$ is the optimal POT reorder interval for a fictitious single retailer whose setup cost is $\sum_{i \in S^0 \cap N^0} (\theta_i K_0 + K_i)$ and whose holding-cost parameter is $G(S^0 \cap N^0)$. This is the case because by definition of θ_i,

$$\tau^2 = \tau^2_{\min}(N) = \frac{\theta_i K_0 + K_i}{g_i} = \frac{\sum_{i \in S^0 \cap N^0} (\theta_i K_0 + K_i)}{G(S^0 \cap N^0)}, \quad i \in S^0 \cap N^0. \quad (9)$$

This completes the proof. □

According to the core cost allocation proposed in Theorem 2, if all the retailers have identical cost parameters and they differ only in their demand rates, then the major setup cost allocation will be linearly increasing in the demand rate of the retailers within the minimal set, but the retailers outside the minimal set will not pay anything toward this cost. Therefore, we can say that in the proposed core cost allocation, the major setup cost allocation is nondecreasing in the demand rate. The justification for such an allocation is that the frequency that the major setup cost is charged is determined by the retailers with the largest sales volume, and therefore they should pay for it. Indeed, within the minimal set, the major setup cost allocation is in accordance to the pro rata cost allocation schemes that are commonly used in the shipment of bulk carriers. However, it is hard to justify economically the fact that retailers outside the minimal set, even if they order rarely, do not pay anything toward the major setup cost.

In addition, the proposed core cost allocation does not satisfy the economies-of-scale property, which gives benefit to retailers with higher demand rates. Indeed, for the identical cost parameter model discussed above, we claim that the benefit of cooperation within the minimal set tends to decrease as demand increases. To see that, note that if retailer $i \in N^0$ acted alone, it would have paid the whole major setup cost K_0 and would have ordered every τ_i time units, which would have cost it on average $\sqrt{2(K_0 + K) h d_i}$, where K is the minor setup cost and h is the holding-cost rate for all the retailers. By cooperating, this retailer is paying only $\theta_i K_0$ toward the major setup cost and it orders every $T_{\min}(N)$ time units (which is the POT rounding-off of $\tau_{\min}(N)$). Now, by construction of the core cost allocation in Theorem 2, θ_i is chosen such that $T_{\min}(N)$ is the optimal POT reorder interval for retailer $i \in N^0$ when it cooperates. In such a case, retailer i would be charged $\sqrt{2(\theta_i K_0 + K) h d_i}$ per time unit, which is $100\% \sqrt{(\theta_i K_0 + K)/(K_0 + K)}$ of what it would have paid if it deviated and acted alone. As we can see, the savings of cooperation is increasing as θ_i is decreasing, or alternatively as the demand is decreasing. This tendency also holds outside the minimal set where the retailers pay exactly what they would have paid had $K_0 = 0$.

In the next theorem, we prove that the core contains infinitely many allocations. The allocations that we found have similar properties to the allocation proposed in Theorem 2. In particular, if the minimal set is not a singleton, then the retailers outside the minimal set pay nothing toward the major setup cost, and within the minimal set, the retailers with larger demands tend to pay a greater share of K_0 than the ones with smaller demands.

THEOREM 3. *There are infinitely many core allocations.*

PROOF. We claim that without loss of generality we can assume that $\tau_{\min}(N) > T_{\min}(N)/\sqrt{2}$. Otherwise, recall from our discussion in §2 that we scaled the base time unit B to 1. Therefore, a small perturbation of B to $B = 1 + \epsilon$ for a small positive ϵ, and reapplying the algorithm with this new base time unit, results in $\tau_{\min}(N)$, which is not of the form $T_{\min}(N)/\sqrt{2}$, where $T_{\min}(N) = B2^m$ for some integer m.

For the sake of the proof we define the following set of constraints on θ_i, $i \in N^0$:

$$\sum_{i \in N^0} \theta_i = 1,$$
$$\frac{T^2_{\min}(N)}{2} \leqslant \frac{\theta_i K_0 + K_i}{g_i} < 2T^2_{\min}(N), \quad i \in N^0,$$
$$\theta_i \geqslant 0, \quad i \in N^0.$$

The core allocation proposed in Theorem 2 satisfies this set of linear constraints. Moreover, if $|N^0| > 1$, the above set of constraints has an infinite number of solutions as one can construct an infinite number of small perturbations of the θ_i values for $i \in N^0$, proposed in Theorem 2, that satisfy the above set of constraints. The proof that such perturbations are also core allocations follows along the lines of the proof of Theorem 2, except for noting that (9) is replaced by the fact that $T^2_{\min}(N)/2 \leqslant (\theta_i K_0 + K_i)/g_i < 2T^2_{\min}(N)$ implies that

$$\frac{T^2_{\min}(N)}{2} \leqslant \frac{\sum_{i \in S^0 \cap N^0} (\theta_i K_0 + K_i)}{G(S^0 \cap N^0)} < 2T^2_{\min}(N).$$

It remains to prove the theorem for the case where $i^*(N) = 1$.

By the algorithm, the retailers in the minimal set $N^0 = \{1\}$ reorder every $T_{\min}(N)$ time units, where $T_{\min}(N)$ is the POT rounding-off of $\tau_{\min}(N)=\tau_1$, and each retailer $i\notin N^0$ replenishes every T_i' time units, where T_i' is the POT rounding-off of τ_i'. As said above, without loss of generality, we can assume that $T_{\min}(N)/\sqrt{2}<\tau_1\leqslant\sqrt{2}T_{\min}(N)$. We allocate the major setup cost as follows: Let $\theta_1=1-\epsilon$ and $\theta_2=\epsilon T_2'/T_{\min}(N)$, for sufficiently small ϵ that satisfies the following set of linear constraints that we denote by (LP(ϵ)):

$$\frac{T^2_{\min}(N)}{2}<\frac{\theta_1K_0+K_1}{g_1}\leqslant 2T^2_{\min}(N),$$

$$\frac{(T_2')^2}{2}\leqslant\frac{\theta_2K_0+K_2}{g_2}<2(T_2')^2,$$

$$\theta_1=1-\epsilon,$$

$$\theta_2=\frac{\epsilon T_2'}{T_{\min}(N)},$$

$$\epsilon\geqslant 0,\quad \theta_1>0.$$

In particular, the solution $\epsilon=0$ is feasible for (LP(ϵ)). In view of the fact that $\tau_1>T_{\min}(N)/\sqrt{2}$ for sufficiently small $\epsilon>0$, the first inequality in (LP(ϵ)) is feasible. Also, in view of the fact that $\tau_2'<\sqrt{2}T_2'$, the second inequality holds too for sufficiently small $\epsilon\geqslant 0$. Thus, the set of feasible solutions of (LP(ϵ)) contains an infinite number of solutions. Accordingly, for each feasible ϵ, we propose the following allocation: Retailer 1 pays

$$x_1=\frac{\theta_1K_0+K_1}{T_{\min}(N)}+g_1T_{\min}(N)=\frac{(1-\epsilon)K_0+K_1}{T_{\min}(N)}+g_1T_{\min}(N),$$

Retailer 2 pays

$$x_2=\frac{\theta_2K_0+K_2}{T_2'}+g_2T_2'=\frac{(\epsilon T_2'/T_{\min}(N))K_0+K_2}{T_2'}+g_2T_2',$$

and retailer i for $i\geqslant 3$ pays

$$x_i=\frac{K_i}{T_i'}+g_iT_i'.$$

The proposed allocation is efficient (or cost sharing), i.e., $\sum_{i=1}^N x_i=v(N)$, because the total major setup cost paid per unit of time is

$$\frac{\theta_1K_0}{T_{\min}(N)}+\frac{\theta_2K_0}{T_2'}=\frac{K_0}{T_{\min}(N)}.$$

To complete the proof, we next prove the stand-alone condition, namely, that

$$\sum_{i\in S}x_i\leqslant v(S),\quad S\subseteq N. \tag{10}$$

Recall that

$$v(S)=\frac{K^0(S^0)}{T_{\min}(S)}+T_{\min}(S)G(S^0)+\sum_{j=i^{*+}(S),\,j\in S}^{n}\left(\frac{K_j}{T_j'}+g_jT_j'\right). \tag{11}$$

We distinguish between four cases toward proving inequality (10). If $1\notin S$ and $2\notin S$, then (10) holds as $\sum_{i\in S}x_i=\sum_{i\in S}(K_i/T_i')+g_iT_i'<v(S)$. In this case, the left-hand side of (10) does not include any share of the major setup cost K_0, and each retailer pays its separate optimal POT policy cost.

If $1\notin S$ and $2\in S$, then $2\in S^0$. Thus,

$$\sum_{i\in S}x_i=\frac{\theta_2K_0+K_2}{T_2'}+g_2T_2'+\sum_{i\in S,\,i\neq 2}\left(\frac{K_i}{T_i'}+g_iT_i'\right).$$

According to (11), any retailer $i\in S\backslash S^0$ is associated with a term identical to x_i. Thus, we need to show that

$$\sum_{i\in S^0}x_i\leqslant\frac{K_0+K_2}{T_{\min}(S)}+g_2T_{\min}(S)+\sum_{i\in S^0\backslash\{2\}}\left(\frac{K_i}{T_{\min}(S)}+g_iT_{\min}(S)\right).$$

We next make the comparison term by term, i.e., we show that

$$x_2\leqslant\frac{K_0+K_2}{T_{\min}(S)}+g_2T_{\min}(S)\quad\text{and}\quad x_i\leqslant\frac{K_i}{T_{\min}(S)}+g_iT_{\min}(S)$$

for $i\in S^0\backslash\{2\}$. The inequality holds for $i\in S^0\backslash\{2\}$, as $K_i/T_i'+g_iT_i'\leqslant K_i/T_{\min}(S)+g_iT_{\min}(S)$. Regarding $i=2$, we need to show that

$$x_2=\frac{\theta_2K_0+K_2}{T_2'}+g_2T_2'=\frac{(\epsilon T_2'/T_{\min}(N))K_0+K_2}{T_2'}+g_2T_2'\leqslant\frac{K_0+K_2}{T_{\min}(S)}+g_2T_{\min}(S).$$

For sufficiently small values of $\epsilon>0$, $\epsilon T_2'/T_{\min}(N)<1$; thus, in the left-hand side of the above inequality, Retailer 2 pays less than K_0 as its share in the major setup cost. In addition, note that according to the second inequality in (LP(ϵ)), T_2' is the optimal POT reorder interval for a fictitious retailer with a setup cost of $(\epsilon T_2'/T_{\min}(N))K_0+K_2$ and holding-cost parameter of g_2. This completes the proof of this case.

Suppose now that $1\in S$. From the algorithm's properties and the fact that $i^*(N)=1$, it follows that $S^0=\{1\}$, and therefore $T_{\min}(N)=T_{\min}(S)$. We start with the case where $2\notin S$. In this case, it is sufficient to show that

$$\frac{(1-\epsilon)K_0+K_1}{T_{\min}(N)}+g_1T_{\min}(N)\leqslant\frac{K_0+K_1}{T_{\min}(S)}+g_1T_{\min}(S),$$

which follows directly. Suppose now that $2\in S$. Then, it remains to show that

$$\frac{(1-\epsilon)K_0+K_1}{T_{\min}(N)}+g_1T_{\min}(N)+\frac{(\epsilon T_2'/T_{\min}(N))K_0+K_2}{T_2'}+g_2T_2'\leqslant\frac{K_0+K_1}{T_{\min}(S)}+g_1T_{\min}(S)+\frac{K_2}{T_2'}+g_2T_2'.$$

In fact, this inequality holds as an equality because $T_{\min}(N)=T_{\min}(S)$. $\square$

As discussed above, the proposed core allocations charge more toward the major setup cost from retailers that need to order more frequently, but they suffer from three drawbacks. The first is that the set of retailers $N\backslash N^0$ seems not to pay its fair share of the major setup cost. In fact, as it pays (almost) nothing against K_0, it can be seen as a set of free riders. The second is the lack of the economies-of-scale property, as we have demonstrated for the identical cost parameters case. Another drawback is the fact that each retailer pays the direct holding costs it inflicts under the prescribed policy. This can be seen as unfair by the retailers in N^0 because their actual replenishment interval $T_{\min}(N)$ might be significantly larger than their unconstrained interval. As a result, the retailers in N^0 may pay a greater holding cost than what they would have paid had $K_0 = 0$. Still, our results ensure that no subset of retailers will gain by deviating from the coalition and acting independently.

In the introduction, we described a cost allocation scheme in which each retailer pays his direct costs (minor setup cost plus his holding costs), while the major setup cost for each order is shared evenly between all the retailers that are participating in this joint order. We next demonstrate, by using Example 1, that this is not a core allocation.

Example 1 (continued). Under the POT optimal policy, which costs on average 8.25, the two retailers order together every eight units of time. If Retailer 2 then pays half of the major setup cost of 15, he ends up paying $(7.5 + 1)/8 + 8/64 = 1.1875$ per unit of time, where Retailer 1 pays $8.25 - 1.1875 = 7.0625$. Retailer 2 will be better off to break away and to use his own EOQ policy of ordering every 32 time units, as his cost will be reduced to one per unit of time. In summary, using the optimal POT joint replenishment policy and sharing the major setup cost evenly between the two retailers (while each pays his own direct costs) is not a core allocation.

5. Conclusions

In this research, we study the cost allocation problem in the joint replenishment model with first-order interaction. We show that a cooperation among retailers in a supply chain with the above cost structure is beneficial not only from the whole system point of view, but also for each of the retailers separately. In our analysis, we follow the cost allocation fairness concept of balancedness developed in game theory and apply it to the joint replenishment model discussed here.

We presented a family of infinitely many cost allocation schemes in the core of the game induced by the above joint replenishment model. According to the schemes proposed, some of the retailers (i.e., those who are not members of the minimal set) pay almost nothing toward the major setup cost expense. These allocation schemes, which are fair theoretically, may cause some of the retailers to feel unhappy: In particular, the retailers in the minimal set may argue that they do not want to cooperate with the other retailers because they pay only for their optimal POT policy as if no major setup cost existed. Psychologically, it is clear that this coalition will not resist for the long run as the minimal set will probably break away at some point from the grand coalition, a step that will not worsen their monetary status. One important question is whether there exist core cost allocations of a different structure according to which all retailers (on top of their minor setup costs and holding costs) pay a reasonable share of the major setup cost expense.

We hope that this paper will trigger more research on cost allocation in joint replenishment systems. Future research should deal with other joint replenishment models. For example, the warehouse may also be a party in the coalition, as in Roundy (1985), where in addition to the retailers' minor setup costs and inventory holding costs, the model includes a warehouse setup cost that is charged each time the warehouse places an order from the supplier, as well as the inventory holding costs at the warehouse. The objective in this problem is to minimize the systemwide average costs. The author proposes a POT policy that costs at most 6% (2%) of the optimal average cost depending on whether the base time unit is given or optimized. However, no one has yet considered the issue of cost allocation among the warehouse and the retailers in this problem. Another question that is of particular interest is whether it is possible to generalize this type of results (e.g., nonempty core) to any joint replenishment system with submodular setup cost function, for which Federgruen and Zheng (1992) construct an optimal POT policy that comes within 6% or 2% of a lower bound on the optimal average-time total cost, depending on whether the base time unit is fixed or variable. Unfortunately, our concavity proof of the characteristic cost function is tailored to the specific model considered here. Thus, other techniques should be developed for the more general case, if it is true. Future research should also address the joint replenishment problem in more realistic settings, in particular, when demands are nonstationary.

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