Dynamic Blockholder Incentives: Liquidity and Reputation*

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Abstract

We study strategic trading by a blockholder who can intervene over time to influence the firm's cash flows. We consider the impact of asymmetric information on the incentives of the blockholder to trade, and study when information asymmetry increases blockholder ownership and leads to greater firm value. Asymmetric information reduces the speed of blockholder trading if private information is sufficiently persistent, but can increase it otherwise. We study how the presence of liquidity shocks, leading to a noisy equilibrium, creates Rachet effects whereby the blockholder's (endogenous) trading plans induce him to distort the firm cash flows to manipulate the stock price.

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1 Introduction

Blockholders play a prominent role in capital markets. They can be institutional investors (e.g., hedge funds, pensions funds, venture capitalists) or wealthy individuals (e.g., firm founders or senior management). They monitor firms and promote changes through various channels (e.g., negotiations with management, proxy fights, etc). These activities are personally costly to the blockholder, and small shareholders free ride on their effort. A blockholder thus faces a trade-off: he can mitigate free riding and enhance his incentive to monitor the firm by owning a large stake, but, by doing so, he compromises his portfolio diversification needs.[1]

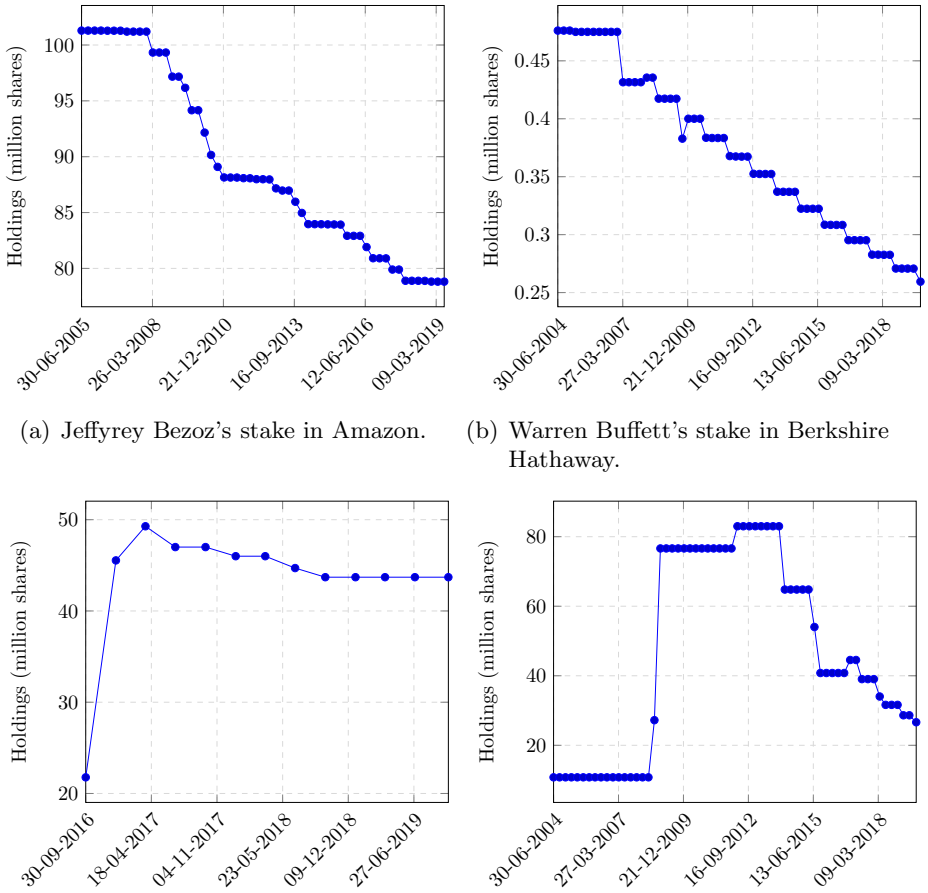
DeMarzo and Urošević (2006) study the dynamics of this trade-off under symmetric information, and prove that a blockholder's stake shrinks over time towards a fully diversified portfolio. In the long-run, a blockholder holds a small stake, thus facing weak incentives to monitor the firm, as if he did not a play any governance role. Under symmetric information — one might conclude — blockholders are bound to play a very limited governance role.

We study strategic trading when a blockholder has access to private information and can affect the firm's cash flows. Specifically, we investigate the impact of asymmetric information on the dynamics of blockholder stakes, firm productivity, and stock prices. We show that, under information asymmetry, a risk-averse blockholder tends to hold a relatively large stake, effectively holding an undiversified portfolio, in contrast with the results arising under symmetric information. We demonstrate that under plausible conditions, stock prices are higher in the presence of asymmetric information.

Figure 1 exhibits four real world examples of the problem we investigate here. The top panels show ownership dynamics for two founders — e.g., Jeff Bezos and Warren Buffet – where we see that founders typically divest their stakes over time, but tend to do it slowly. The bottom panels show ownership dynamics for two large funds — e.g., Berkshire Hathaway and Trian. These examples capture situations whereby an activist fund, such as Trian, learns about an opportunity to create value in a target firm, such as Wendy's, and increases its stake over time to profit from the opportunity, and proceeds to unwind it afterwards. To be effective, Trian needs to enter (and exit) the firm carefully to avoid triggering large price reactions that could threaten its profit opportunity.

Our baseline model builds on DeMarzo and Urošević (2006) but allows for time-varying blockholder ability and asymmetric information. Specifically, we consider a dynamic model of trading between a large investor (or blockholder) and a competitive fringe of small investors (henceforth, the market). In each period, the blockholder can trade and make costly decisions to influence the

[1] These trade-offs have been long identified by corporate governance scholars and practitioners at least going back to the work by Berle and Means (1932), Alchian and Demsetz (1972), and Jensen and Meckling (1976).



(a) Jeffyrey Bezoz's stake in Amazon.

(b) Warren Buffett's stake in Berkshire Hathaway.

(c) Berkshire Hathaway's stake in American Airlines.

(d) Trian Fund Management's stake in Wendy's.

Figure 1: Examples of Blockholder Ownership Dynamics. Source: S&P Capital IQ.

firm's cash flows. Crucially, the blockholder cannot commit to holding a large stake, and trades continuously based on his private information and hedging needs. The main source of private information is the blockholder's ability to influence the firm's cash flows, which varies over time. In other words, there is information asymmetry regarding the blockholder's ability to add (or extract) value to the firm.

In our baseline model, trading is fully revealing and the blockholder's trading choices are affected by signaling incentives similar to those in Leland and Pyle (1977) (thus, our model also contributes to the literature on signaling by considering a dynamic model of ownership). The market does not observe blockholder ability, but assesses it based on the blockholder's trading history (and the firm cash flows). In equilibrium, the blockholder faces a relatively illiquid market because his trading is

informative and, thus, has a price impact. In effect, when the market observes that the blockholder is buying shares, it anticipates stronger and more effective monitoring, hence higher future cash flows. This, naturally boosts the stock price. On the other hand, the blockholder, anticipating his price impact, may trade slowly to benefit from the value that he will create via stronger monitoring.

We start off by considering, as a benchmark, trading under symmetric information. In this case, a positive ability shock triggers an immediate jump in the stock price, as the market anticipates more effective and intense monitoring. However, the blockholder responds by selling shares, for diversification reasons. By reducing his stake, the blockholder weakens his own incentive to "work." Under symmetric information, the blockholder's trading is characterized by Coasian dynamicsbecause the blockholder is unable to exploit his market power due to lack of commitment , as in DeMarzo and Urošević (2006).[2] Hence, the blockholder sells shares towards a more diversified portfolio. As his stake shrinks, the blockholder is less able to internalize the cash flow impact of weaker monitoring, and this process continues over time until the blockholder portfolio is fully diversified.

The introduction of asymmetric information qualitatively changes the dynamics of trading and asset prices. In response to a positive ability shock, the blockholder now buys shares (given the initial underpricing of the stock) and holds them while the shock persists. The Coase conjecture no longer holds: due to signaling effects, the blockholder's trading has a price impact; when the blockholder buys shares, the market updates its beliefs about firm profitability upwards leading to a stock price increase. In turn, this illiquidity introduces a wedge between the blockholder's marginal valuation and that of the market.

We find that under asymmetric information, the blockholder's portfolio adjustments can be quicker than under symmetric information. When private information is sufficiently persistent, the anticipated price impact moderates the blockholder's trading speed, thus providing an implicit commitment device that induces the blockholder to retain his shares for longer. Surprisingly, when ability shocks are transitory, the blockholder trades faster under asymmetric information, despite the illiquidity he faces.

The presence of information asymmetry can have long-run consequences on the firm's ownership structure and its productivity. In particular, when there is a risk premium associated with the blockholder's private information, the higher cost of having the market absorb the (private informa-

[2]This lack of commitment was first studied by Coase (1972). The paradox asserts that a monopolist selling durable goods (e.g., houses) effectively competes against his future sales. Anticipating this form of competition, the monopolist would choose to charge a competitive price in the first place. The monopolist's inability to commit to not selling all his inventory, so to exploit his market power, would eliminate his monopoly rents, in a dynamic context. In addition to the work by DeMarzo and Urošević (2006), such a commitment problem in models with large shareholders has been studied by Kihlstrom (2000), and Gorton et al. (2014).

tion) risk distorts risk allocation, and the blockholder holds a larger stake than under symmetric information. In that case, information asymmetry brings about stronger monitoring and higher firm productivity. Though information asymmetry often leads to greater cash flow volatility, under plausible conditions, it yields a higher stock price in the long-run (on average). By contrast, when the blockholder's private information commands no risk premium, either because the market is risk neutral or because (private information) risk can be diversified away, then asymmetric information only has transitory effects on the firm's ownership structure, but in the long-run, the blockholder's portfolio converges to full diversification, as under symmetric information.

The literature has examined the role of liquidity in facilitating blockholder activism, but the analysis has focused on static settings. Two opposing arguments have been advanced: While liquidity makes it easier for the blockholder to build his block (Coffee (1991)) thereby facilitating blockholder monitoring, it, on the other hand, makes it easier for the blockholder to unwind his position, thereby decreasing the duration of blockholder monitoring (Maug, 1998; Kyle and Vila, 1991; Back et al., 2018). It is thus unclear whether liquidity promotes activism or weakens it, as the answer, in prior literature, seems determined by assumptions about the blockholder's initial stake. By considering a dynamic setting, we are able to answer this question. We find that under information asymmetry, liquidity tends to be low. While, this may slow down the speed at which the blockholder builds his stake, relative to the symmetric information case, it leads in the long-run to a larger stake, hence stronger monitoring. In a nutshell: the illiquidity caused by information asymmetry can be detrimental in the short-run but plays a favorable role in the long-run by inducing higher blockholder monitoring (or activism).

We extend the model to incorporate unobservable liquidity shocks. In this case, the equilibrium is not fully revealing as the market can't tease apart whether the blockholder's trading is motivated by private information about the firm or his own liquidity needs. The market uses two signals to learn about the firm's fundamentals, the evolution of cash flows, and the blockholder's trading behavior. Private information determines a target for the blockholder stake. Given his price impact, the blockholder adjusts his portfolio slowly towards his target stake. At the same time, the blockholder begins to distort cash flows, by altering effort, to manipulate market beliefs and, ultimately, the stock price. For example, when the blockholder position is below its target, so the blockholder intends to buy shares, he reduces his effort to depress cash flows, which in turn leads to a lower stock price. The incentive to distort effort is related to the Ratchet effect in the literature on career concerns (Holmström, 1999). However, unlike in the career concerns literature, the incentive to over and under provide effort is endogenous and jointly determined with the blockholder trading strategy.

In addition, by introducing unobservable persistent liquidity shocks into a dynamic trading

model with asymmetric information, our paper also makes a methodological contribution. Our model cannot be solved using standard techniques because the market perfectly observes the trading rate, which is a linear combination of two mean reverting processes. Thus, conditional on the observed trading rate, liquidity and ability shocks are collinear. Using techniques from the literature on singular filtering, we transform the original two dimensional filtering problem into a one dimensional problem that can be analyzed using standard techniques. In the new one-dimensional filtering problem, the market adjusts its beliefs based on the changes in the trading rate instead of the level of the trading rate (as is the case in standard models, e.g. Kyle (1985)). This is natural in our setting as the trading rate is driven by two mean reverting processes, so one needs to look at the mean reversion in the trading rate to identify the driving shock. Because beliefs are update based on changes in the trading rate, the impact that today's order has on beliefs depends on yesterday's order, which means that the blockholder's incentive to deviate from the equilibrium trading strategy is affected by previous deviations. Thus, it is not enough to restrict attention to local incentive compatibility constraints to construct an equilibrium, and we need to consider the impact of global deviations. This problem of "double deviations" is similar to the one in the literature on dynamic contracts and games with persistent private information that relies on the "first order approach" (DeMarzo and Sannikov, 2016; He et al., 2017; Cisternas, 2017; Marinovic and Varas, 2019). Following ideas from this literature, we solve for the equilibrium considering local incentive compatibility conditions, and then verifying global optimality by constructing an upper bound to the blockholder's off-path continuation payoff.

Literature The most closely related papers are Huddart (1993), Admati et al. (1994) and DeMarzo and Urošević (2006), who study the incentives of large shareholders to monitor a firm. They emphasize the blockholder's lack of commitment and free riding problem, and highlight the tension between optimal risk-sharing and monitoring incentives, which require concentrated ownership. Our model is based on DeMarzo and Urošević (2006). Our main contribution relative to DeMarzo and Urošević (2006) is to allow for information asymmetry between the blockholder and small investors.

Although blockholders may add value through monitoring, Admati et al. (1994) and DeMarzo and Urošević (2006) show that large blocks are unstable because, in the absence of commitment, a blockholder would tend to reduce his stake over time to decrease his risk exposure. One policy implication of these models is that corporate governance could be improved if blockholders are subsidized to hold large blocks.

In a static setting, Leland and Pyle (1977) shows that, in the presence of asymmetric information, a risk-averse entrepreneur will retain ownership to signal that the firm value is high. This

suggests that asymmetric information might provide an endogenous commitment device for the blockholder to hold his stake for a longer period of time.

Our model features multiple equilibria. This arises in our model from the presence of feedback effects (see e.g., Bond and Eraslan (2010); Bond et al. (2012) between the blockholder's actions and the market beliefs, similar to Edmans et al. (2015). Specifically, there is a complementarity between the amount of effort the blockholder wants to exert at any given point, and the sensitivity of the price to the blockholder's stake (i.e., if the market believes that the blockholder will sell his block quickly, the price becomes less sensitive to blockholder stake, and this in turn induces the agent to trade quickly).

More broadly, our paper belongs to the corporate governance strand that looks at the real effect of blockholders and activist investors. This literature is surveyed in Becht, Bolton, and Röell (2003) and Edmans and Holderness (2017). Starting with Hirschman (1970), the literature on corporate governance has looked at how investors can affect corporate decision by voice (direct intervention) or exit (showing their discontent by selling their shares). Admati and Pfleiderer (2009) and Edmans (2009) show that an investor can intervene in the corporation by exiting when they disagree with the firm's management. The key assumption in these models is that the manager's compensation is tied to the price of the company, so the manager is hurt when selling pressures bring the price down.

Our paper also belongs to the literature studying the impact of liquidity on investor intervention. A key issue in this literature is that, when the firm is under-performing, blockholder may have incentives to sell (cut and run) instead of bearing the cost of interventions. For this reason, it has been argued that market liquidity might harm corporate governance (Coffee, 1991). For example, motivated by this idea, the European Union agreed to implement a transaction tax in September 2016. This trade-off between governance and liquidity has been formally analyzed by Kahn and Winton (1998), Noe (2002) and Faure-Grimaud and Gromb (2004). A counterargument to the thesis in (Coffee, 1991) is that liquidity might reduce the free riding problem identified by Grossman and Hart (1980) and Shleifer and Vishny (1986). By facilitating the creation of a large block in the first place, liquidity can actually strengthen the firm's corporate governance. These argument is formalized by Kyle and Vila (1991), in the contexts of takeovers, and Maug (1998) in the contexts of investor activism.

Most of these models are static in nature, and thus silent about the effect of future trading, identified by Admati et al. (1994) and DeMarzo and Urošević (2006). Our paper contributes to the literature on dynamic trading under information asymmetry (see e.g., Bond and Zhong (2016); Kyle (1985); Kyle et al. (2017)). Recently, Back et al. (2018) analyzed many of these issues in a dynamic setting. They consider a setting similar to Kyle (1985) in which an informed trader

has private information about his initial stock holdings, and can exert costly effort to increase the firm value before it becomes known. Surprisingly, and in contrast to Kyle (1985), they find that the relation between efficiency and liquidity is ambiguous and depends on model parameters. Because liquidity and intervention are simultaneously determined, more noise trading can increase the information asymmetry about the activist's intentions and thus decrease liquidity. Unlike Back et al. (2018), we consider a setting in which intervention is continuous (rather than one-off), the block size is observable, and there is asymmetric information about the blockholder's ability. Also our setting allows for risk aversion, which introduces a trade-off between monitoring, which requires large blocks, and diversification. Moreover, our setting with risk aversion allow us to explore the asset pricing implications of activism.

Gomes (2000) also studies a reputation game, with two types of manager/owners, who differ in terms of their cost of effort. In Gomes (2000) the manager effort is observable. He shows how reputation effects moderate the insider's incentive to expropriate minority shareholders. Unlike Gomes, we allow for hidden effort and time-varying private information. Moreover, our main focus is not the effect of reputation on managerial incentives but rather to show how price impact due to asymmetric information can reduce the commitment problem referenced above and its asset pricing implications.

Finally, there is a relatively small literature in asset pricing that looks at the asset pricing implications of agency frictions in general equilibrium settings. The main lesson from this literature is that, by distorting productive decisions, agency frictions affect the volatility of cash-flows and the overall risk premium. For example, Gorton et al. (2014) considers a Lucas-tree economy, in which the output is determined by the effort of a manager who's compensation depends on output and who can trade the shares of the asset. They show that depending on the risk aversion of the manager, trading by the manager can lead to more or less volatile cash flows an risk premium. Albuquerue and Wang (2008) study the effect of investor protection on welfare and asset pricing in a general equilibrium model with production. They show that weaker investor protection increase agency costs, which lead to over-investment, more volatile cash-flows and larger risk premium.

2 Model

We study the behavior of a large investor (henceforth, blockholder) who can both trade a firm's stock and make costly decisions that affect the firm's cash flows.

There is a singly risky asset/firm. There is a continuum of small investors who trade but can't influence the firm's cash flows. All agents in the economy maximize expected utility and have CARA preferences. Hence, as DeMarzo and Urošević (2006) we can aggregate the competitive

investors into a single, aggregate investor with risk aversion parameter γ_M.

Time t is continuous and the horizon is infinite. There is a single firm in unit supply with a cumulative cash flow process $(D_t)_{t\geq 0}$ evolving as

$$dD_t = (\mu_D + a_t)dt + \sigma_D dB_t^D,$$

where a_t is the blockholder's action and $(B_t^D)_{t\geq 0}$ is a standard Brownian motion. The cash flow dD_t is publicly observable but a_t is not. The market thus faces a moral hazard problem. Without loss of generality, we assume that the realized cash flows are paid to shareholders in each period, and interpret dD_t as the firm's dividends (or cash flows).

We refer to a_t as effort but interpret it broadly as any action of the blockholder that affects the firm's cash flows. The blockholder's effort produces thus an externality on the firm's cash flows. When $a_t > 0$, the externality is positive; that is blockholder effort increases the cash flow. We allow $a_t < 0$, in which case a_t represents the blockholder's rent extraction. We are agnostic as to the source of the blockholder externality. In the case of an external investor, one can think of a_t as the blockholder's monitoring — which disciplines managers and mitigates agency conflicts— or as the influence the blockholder exerts on the firm's management (as in Admati et al. (1994)). Examples of a_t include public criticism of management or launching a proxy fight, advising management on strategy, figuring out how to vote on proxy contest launched by others or not taking private benefits for himself. In the case of a CEO or the founder of a company, a_t can represent effort or a reduction of private benefits that increases the productivity of the firm.

The blockholder privately bears the cost of effort.[3] The small shareholders free ride on the blockholder's effort. The blockholder's cost of effort is given by

$$\Phi(a_t, \zeta_t) = \phi a_t^2 - \psi \zeta_t a_t,$$

Hence, the cost of effort depends on two variables: effort a_t, and ability ζ_t.[4] Broadly, the term $\psi\zeta_t a_t$ captures private benefits that the blockholder receives from his effort to influence the firm. Cross sectional differences in ability and preferences are realistic: Cronqvist and Fahlenbrach (2008) find significant blockholder fixed effects in investment, financial, and executive compensation policies.[5]

[3]Bill Ackman, a well known hedge fund activist, asserts "Shareholder activism is extremely time-consuming, expensive and a drain on an investment firm's resources." See "For Activist Investors, a Wide Reporting Window", The New York Times, May 19, 2014.

[4]It is natural to think that blockholder ability depends on the blockholder's holdings, X. The model does not qualitatively change if the cost function includes a term $-\chi aX$, but to simplify the exposition we don't include it.

[5]The Economist analyzed the 50 largest activist positions in America since 2009 and found that on average, profits, capital investment, and R&D have risen. See "Shareholder activism Capitalism's unlikely heroes", The Economist, February, 2015.

Effort is unobservable to the market. Furthermore, the blockholder privately observes his ability $(\zeta_t)_{t\geq 0}$. Ability is random but persistent. In particular, it evolves according to a mean reverting process

$$d\zeta_t = -\kappa\zeta_t dt + \sigma_\zeta dB_t^\zeta,$$

where $(B_t^\zeta)_{t\geq 0}$ is a Brownian motion independent of $(B_t^D)_{t\geq 0}$. The speed of mean reversion is thus captured by κ. When κ is small, ability shocks are highly persistent.

Ability $(\zeta_t)_{t\geq 0}$ is a stationary Gaussian process

$$\mathbb{E}[\zeta_t] = 0 \text{ and } \text{Cov}[\zeta_t, \zeta_s] = \frac{\sigma_\zeta^2}{2\kappa} e^{-\kappa|t-s|}.$$

The variance of the stationary distribution of ζ_t is given by $\bar{\sigma}_\zeta^2 \equiv \sigma_\zeta^2/2\kappa$.

All agents are risk averse and have preferences with constant absolute risk aversion. Specifically, the flow utility of a trader type i is represented by CARA utility function

$$u_i(c) = -\exp\left(-\gamma_i c\right)$$

for $i \in \{L, M\}$ where c is consumption and γ_i is the coefficient of risk aversion of a type i investor. In this context γ_L/γ_M represents the market's risk-bearing capacity.

The information structure is as follows. The blockholder observes the dividend dD_t and his ability ζ_t. Based on this information set, the blockholder chooses effort, consumption/savings c_t and stock holdings X_t, where X_t is the number of shares the blockholder holds at time t.

Competitive investors observe the dividend process D_t as well as the large shareholder's order flow q_t^L. Hence, the competitive investors information set is given by the filtration $\mathcal{F}_t^M = \sigma(D_s, q_s^L | s \leq t)$, while the blockholder's information set is given by the filtration $\mathcal{F}_t^L = \sigma(D_s, q_s^L, \zeta_s | s \leq t)$. Throughout the paper, we use the notation $\mathbb{E}_t^M[\cdot] \equiv \mathbb{E}[\cdot|\mathcal{F}_t^M]$ and $\mathbb{E}_t^L[\cdot] \equiv \mathbb{E}[\cdot|\mathcal{F}_t^L]$, and denote $\hat{\zeta}_t \equiv \mathbb{E}_t^M[\zeta_t]$.

Competitive investors choose a consumption c_t^M and order flow q_t^M strategy adapted to $\mathcal{F}_t^M$. We denote the aggregate holdings of market makers at time t by Y_t. Since the firm is in unit supply the market clearing condition at time t is

$$X_t + Y_t = 1,$$

Hence, the holdings X_t and Y_t, represent the shareholder and competitive investors percentage of ownership, respectively. We follow Kyle, Obizhaeva, and Wang (2017) and consider equilibria with smooth trading in which the blockholder inventory X_t is an absolutely continuous process, so the

market clearing condition requires that at any time t

$$q_t^M + q_t^L = 0.$$

Optimization Program Denote by W_t the savings of a small investor. Given a $\mathcal{F}_t^M$-adapted price process p_t, at any time t, the competitive investor chooses a $\mathcal{F}_t^M$-adapted strategy $(c_t^M, q_t^M)_{t\geq 0}$ to solve the following problem

$$\begin{aligned}
&\max_{c,q^M} \mathbb{E}_t^M \left[\int_0^\infty e^{-r(s-t)} u_M(c_s) ds \right] \\
&\text{subject to} \\
dW_t &= (rW_t - c_t - p_t q_t^M + (\mu_D + a_t) Y_t) dt + \sigma_D Y_t dB_t^D \\
dY_t &= q_t^M dt.
\end{aligned}$$

The second equation captures the market maker's budget constraint. The market maker's savings grow at the interest rate r. The market makers consumes c_t invests $p_t q_t$ in additional shares and receives dD_t as dividends on their existing shares. Observe that because market makers are a competitive fringe they take the price p_t as given; in other words their order flow does not have a price impact.

On the other hand, the blockholder chooses a $\mathcal{F}_t^L$-adapted strategy $(c_t^L, q_t^M, a_t)_{t\geq 0}$ to solve the following problem

$$\begin{aligned}
&\max_{c,q^L,a} \mathbb{E}_t^L \left[\int_t^\infty e^{-r(s-t)} u_L(c_s) ds \right] \\
&\text{subject to} \\
dW_t &= (rW_t - c_t - \Phi(a_t, \zeta_t) - p_t(q_t^L) q_t^L + (\mu_D + a_t) X_t) dt + X_t \sigma_D dB_t^D \\
dX_t &= q_t^L dt.
\end{aligned}$$

The blockholder chooses effort a_t, consumption c_t, and an order flow q_t^L. The blockholder is privately informed about ζ_t so, unlike the market, he does not need to form beliefs about ζ_t. Also, the blockholder has market power, hence he takes into consideration the price impact of his order flow q_t^L. In fact, his order flow affects the price for two reasons: because of competition and because it conveys information about his ability ζ_t.

In summary, two things distinguish the problem of the blockholder from that of small investors. First, the blockholder does not take the price as given. Second, the blockholder bears the cost of effort $\Phi(a_t, \zeta_t)$ (More generally, we can think of $\Phi(a_t, \zeta_t)$ as capturing the cost of effort net of the

blockholder's private benefits).

Equilibrium definition An equilibrium is a price process p_t and a profile (q_t^L, q_t^M, a_t) such that q_t^M solves the small investors' portfolio problem, (q_t^L, a_t) solves the blockholder's problem, and the market clearing condition $q_t^L = -q_t^M$ is satisfied.

We consider stationary Markov perfect equilibria in which (p_t, q_t^L, q_t^M, a_t) are affine functions of the three natural state variables $(X_t, \zeta_t, \hat{\zeta}_t)$ where

$$\begin{aligned} q_t^L &= Q_0 - Q_x X_t + Q_\zeta \zeta_t \\ a_t &= A_x X_t + A_\zeta \zeta_t \\ p_t &= P_0 + P_x X_t + P_\zeta \hat{\zeta}_t. \end{aligned}$$

Throughout the paper, we use boldface to denote the coefficient vectors $(\mathbf{Q}, \mathbf{A}, \mathbf{P})$.

3 Competitive Investors' Problem

Small investors choose their portfolios based on their beliefs about the blockholder's ability ζ_t and his trading strategy. In particular, given the conjectured strategy, and the blockholder's inventory X_t, the market makers can invert the order flow of the blockholder q_L to infer the exact value of the ability ζ_t. Hence, the evolution of the market makers' belief is given by

$$d\hat{\zeta}_t = -\kappa \hat{\zeta}_t dt + \sigma_\zeta dB_t^\zeta.$$

As usual in Gaussian linear quadratic models with CARA preferences (e.g., see Vayanos and Woolley (2013)), we conjecture and then verify that the value function takes the form

$$J(W, Y, X, \hat{\zeta}) = -\exp\left(-r\gamma_M (W + H(Y, X, \hat{\zeta}))\right)/r,$$

where the function H is the certainty equivalent of a market maker and satisfies the following HJB equation:[6]

$$rH = \max_q \ (\mu_D + A_x X + A_\zeta \hat{\zeta})Y - p(X,\hat{\zeta})q - \frac{1}{2} r\gamma_M \left(Y^2\sigma_D^2 + \sigma_\zeta^2 H_\zeta^2\right) \\ qH_y + (Q_0 - Q_x X + Q_\zeta \hat{\zeta})H_x - \kappa\hat{\zeta}H_\zeta + \frac{1}{2}\sigma_\zeta^2 H_{\zeta\zeta} \quad (1)$$

Taking the first order condition for q we get

$$p(X,\hat{\zeta}) = H_y. \quad (2)$$

This condition states that for the market maker to be willing to trade, the price must equal the marginal impact of an additional share on the market maker's certainty equivalent, given his conjecture about ζ_t and the strategy that the blockholder is expected to follow in the future. The market maker computes the firm value, given his belief $\hat{\zeta}_t$ by projecting the trading strategy that the blockholder will adopt and the impact this will have on the firm's future cash flows. The certainty equivalent is given by the quadratic function

$$H(Y,X,\hat{\zeta}) = h_y Y + h_{yx} XY + h_{y\zeta} Y\hat{\zeta} + h_{yy} Y^2,$$

where the coefficient are provided in Lemma A.1 in the appendix. Since the market clearing condition requires $X+Y=1$, the price is given by $p(X,\hat{\zeta}) = H_y(1-X,X,\hat{\zeta})$. Matching coefficients we obtain the coefficients of the price function.

Lemma 1. *Given coefficients* $(\mathbf{Q},\mathbf{A})$*, the coefficients* $\mathbf{P}$ *of the price function are*

$$P_0 = \frac{\mu_D}{r} + \frac{A_x Q_0}{r\,(r+Q_x)} - \gamma_M \left[\sigma_D^2 + \frac{\sigma_\zeta^2}{(r+\kappa)^2}\left(A_\zeta + A_x\frac{Q_\zeta}{r+Q_x}\right)^2\right] \quad (3a)$$

$$P_x = \frac{A_x}{r+Q_x} + \gamma_M \left[\sigma_D^2 + \frac{\sigma_\zeta^2}{(r+\kappa)^2}\left(A_\zeta + A_x\frac{Q_\zeta}{r+Q_x}\right)^2\right] \quad (3b)$$

$$P_\zeta = \frac{1}{r+\kappa}\left(A_\zeta + A_x\frac{Q_\zeta}{r+Q_x}\right). \quad (3c)$$

Observe that these conditions hold both when ζ_t is unobservable and when ζ_t is public information. The price function is determined by competition among small investors: given their beliefs

[6] This equation is not formally an HJB equation but it is derived from the HJB equation due to the term $\frac{1}{2}r\gamma_M\left(Y^2\sigma_D^2 + \sigma_\zeta^2 H_\zeta^2\right)$. That being said, hereafter, we refer to it as the HJB equation with some abuse of terminology.

about the state and the blockholder strategy, small investors break even for any order that the blockholder may place. The price is sensitive to the stake of the blockholder, X_t, for two reasons: first, the impact of the blockholder on the firm's productivity depends on the blockholder stake. Second, the larger the blockholder stake, the lesser risk the market absorbs, which lowers the risk premium.

4 Benchmark: Symmetric Information

Before solving the blockholder's problem and characterizing the equilibrium, we study the case when ζ_t is observable using as a starting point the solution to the market makers' problem characterized in the previous section. As a special case, we provide the solution when the ability of the blockholder is irrelevant $\psi = 0$, which corresponds to the setting in DeMarzo and Urošević (2006).

When ζ_t is observable the market does not need to form beliefs about ζ_t and, for that reason, the price only depends on the holding X_t but the order flow q_t^L is irrelevant. As before, we consider a linear equilibrium with the following structure:

$$\begin{aligned} q_t &= Q_0^o - Q_x^o X_t + Q_\zeta^o \zeta_t \\ a_t &= A_x X_t + A_\zeta \zeta_t \\ p_t &= P_0^o + P_x^o X_t + P_\zeta^o \zeta_t. \end{aligned}$$

The market makers' problem was characterized in the previous section. The problem of the blockholder is

$$\begin{aligned} &\max_{c,q,a} \ \mathbb{E}_t^L \left[\int_t^\infty e^{-r(s-t)} u_L(c_s) ds \right] \\ &\text{subject to} \\ &dW_t = (rW_t - c_t - \Phi(a_t, \zeta_t) - p(X_t, \zeta_t) q_t + (\mu_D + a_t) X_t) dt + X_t \sigma_D dB_t^D \\ &dX_t = q_t dt. \end{aligned}$$

One can verify that the value function of the blockholder takes the form

$$V(W, X, \zeta) = -\exp\left(-r\gamma_L (W + G^o(X, \zeta))\right)/r,$$

where the certainty equivalent G satisfies the HJB equation:

$$rG^o = \max_{q,a} (\mu_D + a)X - \phi a^2 + \psi\zeta a - p(X,\zeta)q - \frac{1}{2}r\gamma_L\left(\sigma_D^2 X^2 + \sigma_\zeta^2 (G_\zeta^o)^2\right) + qG_x^o - \kappa\zeta G_\zeta^o + \frac{1}{2}\sigma_\zeta^2 G_{\zeta\zeta}^o \quad (4)$$

Taking the first order conditions, yields

$$a = \frac{\psi\zeta + X}{2\phi} \quad (5a)$$

$$p\,(X,\zeta) = G_x^o \quad (5b)$$

Condition (5a) states that the blockholder effort is a linear function of the blockholder ability and his holdings. This is intuitive: the blockholder exerts more effort when he is more productive. Also, the blockholder exerts more effort when he owns a larger stake, since he internalizes more the benefits of his effort. Put differently, the free riding problem is milder when the blockholder's stake is larger.

Condition 5b says that the price must equal the marginal value of a share to the blockholder. Because of competition the price also equals the marginal value to a market maker, H_y. Hence, when ability is observable, trading is characterized by Coasian dynamics: At each point, the blockholder trades until his marginal valuation equals the price, despite having market power. Trade can be smooth, but at any point the blockholder effectively trades at a price that equals his marginal valuation, as predicted by the Coase conjecture.

As before, we conjecture and verify that the certainty equivalent is a quadratic function of X and ζ. The coefficients are provided in Lemma A.2 in the appendix. There are two solutions to the polynomial describing the equilibrium, which correspond to two different equilibria, but one of them dominates the other in terms of the blockholder's certainty equivalent.

The next step is to find expressions for the coefficients of the trading strategy, $\mathbf{Q}$. We know that $P = G_X^o$, which must coincide with the coefficients in Lemma 1. Matching coefficients, we obtain a system of equations that allows to solve for the trading strategy coefficients, $\mathbf{Q}$. We have the following Proposition:

Proposition 1. *Let*

$$\nu_+ \equiv \left(r + \sqrt{(r+2\kappa)^2 - 2\frac{\psi^2}{\phi} r\gamma_L \sigma_\zeta^2} \right)^{-1}$$
$$\nu_- \equiv \left(r - \sqrt{(r+2\kappa)^2 - 2\frac{\psi^2}{\phi} r\gamma_L \sigma_\zeta^2} \right)^{-1}.$$

If

$$\frac{1}{2r\phi} > (\gamma_L + \gamma_M)\left(\sigma_D^2 + \frac{\psi^2}{\phi^2}\nu_+^2 \sigma_\zeta^2 \right),$$

then there is a Markov Perfect Equilibrium with observable shocks such that the coefficients of the blockholder trading strategy are

$$Q_0^o = \frac{r^2\gamma_M\left(\sigma_D^2 + \frac{\psi^2}{\phi^2}\sigma_\zeta^2\nu_+^2\right)}{(2\phi)^{-1} - r\left(\gamma_L+\gamma_M\right)\left(\sigma_D^2 + \frac{\psi^2}{\phi^2}\nu_+^2\sigma_\zeta^2\right)} \tag{6a}$$

$$Q_x^o = \frac{r^2\phi\left(\gamma_L+\gamma_M\right)\left(\sigma_D^2 + \frac{\psi^2}{\phi^2}\nu_+^2\sigma_\zeta^2\right)}{(2\phi)^{-1} - r\left(\gamma_L+\gamma_M\right)\left(\sigma_D^2 + \frac{\psi^2}{\phi^2}\nu_+^2\sigma_\zeta^2\right)} \tag{6b}$$

$$Q_\zeta^o = \frac{r\left(\frac{\psi^2}{\phi^2}(r+\kappa)\nu_+^2 - \frac{\psi}{2\phi}\right)}{(2\phi)^{-1} - r\left(\gamma_L+\gamma_M\right)\left(\sigma_D^2 + \frac{\psi^2}{\phi^2}\nu_+^2\sigma_\zeta^2\right)}, \tag{6c}$$

If

$$\frac{1}{2r\phi} > (\gamma_L + \gamma_M)\left(\sigma_D^2 + \frac{\psi^2}{\phi^2}\nu_-^2 \sigma_\zeta^2 \right),$$

there is a second equilibrium with coefficients given by (6a)-(6c) *but* ν_- *in place of* ν_+.

We show in the appendix, that the block-holder payoff is always higher in the first equilibrium. Hence, hereafter we focus on the equilibrium with ν_+ as our benchmark case. That being said, the qualitative results presented below hold regardless of the equilibrium considered as a benchmark. In particular, it can be verified that in either equilibrium $Q_\zeta^o < 0$. This means that a positive shock to the blockholder's ability induces the blockholder to sell shares. To understand this result, notice that ζ_t could be interpreted as an endowment shock that increases the exposure of the blockholder to the firm's dividends, $\mu_D + a_t$ (Of course, it's not merely an endowment shock since it also has an impact on the firm's cash flows)

Under CARA preferences, risk aversion induces the blockholder to sell shares in the face of a

positive ability shock and buy otherwise. This means that the potential productivity benefits associated with the blockholder holding a larger stake do not fully materialize because the blockholder reduces his stake precisely when he is most effective. The lack of commitment on the part of the blockholder explains this result.

In fact, the blockholder tends to hold a diversified portfolio, regardless of his ability to monitor the firm. Indeed, the mean blockholder stake in steady state, $\bar{X}^o_{ss} \equiv Q^o_0/Q^o_x$, is

$$\bar{X}^o_{ss} = \frac{\gamma_M}{\gamma_L + \gamma_M}, \tag{7}$$

which coincides with that in DeMarzo and Urošević (2006). The mean stake of the blockholder depends only on relative risk aversions, but is independent of the intensity of moral hazard problem, as measured by ϕ, which suggests that this case may entail very inefficient levels of effort. Indeed, the blockholder thus holds a stake of the same size as that he would hold if he could not monitor the firm (a_t=0). Of course, his inability to commit is behind this inefficiency.

As a special case, we recover the equilibrium when blockholder ability is constant, which corresponds to the solution in DeMarzo and Urošević (2006). Setting $\psi = 0$, we obtain

$$\begin{aligned} Q_0 &= \frac{r^2 \gamma_M \sigma_D^2}{(2\phi)^{-1} - r\left(\gamma_L + \gamma_M\right)\sigma_D^2} \\ Q_x &= \frac{r^2 \left(\gamma_L + \gamma_M\right)\sigma_D^2}{(2\phi)^{-1} - r\left(\gamma_L + \gamma_M\right)\sigma_D^2} \\ P_x &= \frac{\phi^{-1} - 2r\gamma_L\sigma_D^2}{r}. \end{aligned}$$

Finally, we briefly discuss what happens when the condition $Q^o_x > 0$ is violated. In DeMarzo and Urošević (2006), when this condition is violated, the blockholder jumps immediately to the competitive solution, with $X_t = \bar{X}^o_{ss}$. The same is true in our case although the competitive solution is not constant due to shocks to ζ_t. To illustrate this point, let X^{o*}_t be the target holding defined by the condition that $q_t = 0$ so the blockholder does not trade away of his current position. By definition, we get that

$$X^{o*}_t \equiv \bar{X}^o_{ss} + \frac{Q^o_\zeta}{Q^o_x}\zeta_t,$$

and we can write the evolution equation for X_t as

$$dX^o_t = Q_x(X^{o*}_t - X^o_t)dt.$$

If we consider the the first equilibrium in Proposition 1 we get that Q^o_x and Q^o_ζ diverge to infinity

when the denominator of Q_x^o becomes zero. However, the ratio Q_ζ^o/Q_x^o converges to a finite negative number. In the limit, the blockholder instantly adjust his position to the target and $X_t^o = X_t^{o*}$.

In sum, if shocks are observable, the blockholder adjusts his holdings instantly in response to a shock. This is a consequence of the Coasian dynamics highlighted by DeMarzo and Urošević (2006). We will see that with asymmetric information, that is no longer the case. As we show in the next section, the incentive to signal high or low ability leads the blockholder to refrain from trading fast and generates an equilibrium with smooth trading.

Remark 1. Two aspects of the previous solution are worth noting. First, notice that the mean stationary holdings when $\psi = 0$ is the same as the one when $\psi > 0$ and ζ_t is observable. Hence, time-varying ability may only affect the average long-term stake under information asymmetry. Second, even though in our continuous time formulation the price impact, P_x, and long term stake are the same as the one in DeMarzo and Urošević (2006), the rate of trade is higher. In fact, the rate of trade in (DeMarzo and Urošević, 2006, Equation 24 in p. 797) is

$$Q_x = \frac{r^2 \left(\gamma_L + \gamma_M\right) \sigma_D^2}{(2\phi)^{-1} - r\gamma_L \sigma_D^2}.$$

Both expressions coincide only if the market is risk neutral ($\gamma_M = 0$). The general lack of convergence between the discrete time limit and the continuous time solution arises because in continuous time the order flow does not increase the instantaneous risk exposure of the market (which depends on the residual supply $1 - X_t$), so there is no instantaneous price impact. Consistent with this, the rate of trade is higher than in the discrete time limit.

5 Asymmetric Information

We return to the general case in which the blockholder's ability ζ_t is unobservable. This case poses some challenges. To be able to value the firm shares, the market must infer the evolution of ζ_t because the firm's productivity is linked to ζ_t. The market may infer this based on the two signals available, the firm's cash flows, and the blockholder order flow. In turn, this inference problem creates incentives for the blockholder to manipulate the market beliefs by distorting his trading.

Consider the blockholder's problem. Since X_t and q_t are observable, the market forms its belief $\hat{\zeta}_t$ by inverting the blockholder's trading strategy as follows:

$$\hat{\zeta}(q_t, X_t) = \frac{q_t - Q_0 + Q_x X_t}{Q_\zeta} \tag{8}$$

Substituting $\hat{\zeta}(q_t, X_t)$ in the price function yields

$$p(X_t, \hat{\zeta}_t) = P_0 + P_x X + P_\zeta \hat{\zeta}(q_t, X_t), \tag{9}$$

so the residual supply function faced by the blockholder can be written as

$$R(q_t, X_t) = R_0 + R_x X_t + R_q q_t, \tag{10}$$

where the coefficients satisfy

$$\begin{aligned} R_0 &= P_0 - \frac{P_\zeta}{Q_\zeta} Q_0, \\ R_x &= P_x + \frac{P_\zeta}{Q_\zeta} Q_x, \\ R_q &= \frac{P_\zeta}{Q_\zeta}. \end{aligned}$$

This function captures the price facing the blockholder as a function of his order flow. Unlike the case with observable ability, the price that the blockholder must pay for a share does not depend on ζ_t directly, but only indirectly via the order flow. In general, the more relevant the blockholder ability, as measured by ψ, the more sensitive is the price to the order flow q_t^L. This means that the liquidity faced by the blockholder decreases when ζ_t is unobservable, particularly so when his ability is more relevant to the firm.

We provide the blockholder's problem under information asymmetry as:

$$\begin{aligned} &\max_{c, q^L, a} \ \mathbb{E}_t^L \left[\int_t^\infty e^{-r(s-t)} u_L(c_s) ds \right] \\ &\text{subject to} \\ &dW_t = (rW_t - c_t - \Phi(a_t, \zeta_t) - R(X_t, q_t^L) q_t^L + (\mu_D + a_t) X_t) dt + X_t \sigma_D dB_t^D \\ &dX_t = q_t^L dt \end{aligned}$$

The blockholder faces a similar problem as in the observable case except that, in choosing his trading strategy, he must take into account the signaling effect of his order flow; namely its price effect. As with the market makers, we conjecture that the value function of the blockholder takes the form

$$V(W, X, \zeta) = -\exp\left(-r\gamma_L (W + G(X, \zeta))\right)/r,$$

where the blockholder's certainty equivalent G satisfies the following HJB equation:

$$rG = \max_{q,a} (\mu_D + a)X - \phi a^2 + \psi\zeta a - R(X,q)q - \frac{1}{2}r\gamma_L\left(\sigma_D^2 X^2 + \sigma_\zeta^2 G_\zeta^2\right) + qG_x - \kappa\zeta G_\zeta + \frac{1}{2}\sigma_\zeta^2 G_{\zeta\zeta} \quad (11)$$

Taking the first order conditions, yields the effort and trading strategy of the blockholder:

$$a = \frac{\psi\zeta + X}{2\phi}$$
$$q = \frac{G_x - R_0 - R_x X}{2R_q}.$$

Two observations are in order. First, the effort strategy is myopic. This is due to the fact that cash flows are not informative, conditional on the order flow q_L. In section 6, we generalize the model to a setting where the order flow is not fully revealing, and the blockholder distorts cash flows via effort to affect his reputation and, ultimately, the stock price.

Second, while the stock price is always equal to the market's marginal valuation ($P = H_y$), there is a wedge between the blockholder's marginal valuation G_x and the stock price $R(q, X)$. This wedge is given by the price effect of the blockholder's order flow, R_q. Indeed, we can rewrite the first order condition above as $G_x - R(X,q) = R_q(X,q)q$. Using the fact that $R(X_t, q_t(X_t,\zeta_t)) = P_t(X_t,\zeta_t) = H_y(1 - X_t, X_t, \zeta_t)$ we get

$$q_t = \frac{G_x(X_t,\zeta_t) - H_y(1 - X_t, X_t, \zeta_t)}{R_q}.$$

In contrast to the observable case —in which the blockholder trades at a competitive price— the presence of private information mitigates the blockholder's commitment problem, and moderates his tendency to trade fast, effectively introducing a wedge between the price and the marginal valuation of the blockholder. The lower the market's liquidity, the larger the gap between his marginal valuation and the price he faces.

The second order condition is satisfied if $R_q > 0$, that is, if the residual supply has a positive slope. The next result characterizes the blockholder's certainty equivalent as a quadratic function of the two state variables ζ_t and X_t.

Lemma 2. *The large shareholder's certainty equivalent is given by*

$$G(\zeta, X) = g_0 + g_x X + g_\zeta \zeta + g_{xx} X^2 + g_{\zeta\zeta}\zeta^2 + g_{x\zeta} X\zeta,$$

where the coefficients are given by the solution to equations (A.9a) - (A.9f).

We can then use the first order conditions to obtain the coefficients $\mathbf{Q}$ as given by

$$Q_0 = \frac{g_x - R_0}{2R_q}$$
$$Q_x = \frac{R_x - 2g_{xx}}{2R_q}$$
$$Q_\zeta = \frac{g_{x\zeta}}{2R_q}$$

Using these coefficients together with the equations for R_0 and R_q, we can write the coefficients of the price function in terms of Q_0 and Q_ζ. At the same time, from the solution to the market maker problem the price coefficients also satisfy Equation (3a)-(3c). In equilibrium, both sets of coefficients must coincide. We can derive a system of equations for the coefficients by combining these two equations, and the equations for the coefficients $\mathbf{g}$ in Lemma 2.

Proposition 2. *There exists a linear Markov perfect Bayesian equilibrium with smooth trading if the system of equations* (A.20)-(A.21) *has a positive solution. Given a positive solution* (R_q, Q_x)*, the coefficient* Q_ζ *is given by*

$$Q_\zeta = \frac{\psi}{2(r+\kappa)\phi R_q - (r+Q_x)^{-1}}$$

and the long run mean holding is

$$\bar{X}_{ss} = \frac{\gamma_M}{\omega(R_q, Q_x)\gamma_L + \gamma_M},$$

where $\omega(R_q, Q_x)$ *is given by equation* (A.25) *in the appendix.*

To obtain the equilibrium, we need to solve a system of two polynomial equations. Similar to previous models of trading (Vayanos, 1999, 2001) the main difficulty in finding close form solutions comes from the risk premium associated to the volatility of ζ. To develop intuition, we will consider two limits which can be solved in closed form. First, we consider the case in which both σ_ζ^2 and κ tend to zero at rate such the limit of $\sigma_\zeta^2/2\kappa \to \bar{\sigma}_\zeta^2$ is strictly positive. This captures a situation when ability shocks are small but highly persistent, so the long-run distribution of ability has positive variance. In the second limit, we consider the case in which σ_ζ^2 goes to zero but κ remains fixed, in which case the limit is deterministic. As we discuss later, this limit is equivalent to the case in which we take γ_L, γ_M to zero and σ_D to infinity at a rate such $\gamma_L\sigma_D^2$ and $\gamma_M\sigma_D^2$ are bounded above

zero. This limit captures the case in which the shocks to ζ_t can be diversified so only the dividend shocks dB_t^D command a risk premium.

The next proposition present the case $\sigma_\zeta^2 \to 0$ but $\kappa > 0$, so the stationary distribution is such $\bar{\sigma}_\zeta^2 = 0$. In this case, the limit blockholder stake is the same in both the observable and unobservable case. However, the trajectory is different due to the price impact generated by asymmetric information.

Proposition 3. *Consider the small noise limit $\sigma_\zeta^2 \to 0$, $\kappa > 0$. In the limit, there is a linear Markov perfect Bayesian equilibrium with coefficients*

$$\begin{aligned} R_q &= \frac{\sqrt{\eta^2 + 2r(r+3\kappa)\alpha^2} + \eta}{r\phi(r+3\kappa)(2r+3\kappa)} \\ Q_x &= \frac{\sqrt{\eta^2 + 2r(r+3\kappa)\alpha^2} - \eta}{2\alpha} \\ Q_\zeta &= \frac{1}{2}\frac{\psi(2r+3\kappa)Q_x}{\sqrt{\eta^2 + 2r(r+3\kappa)\alpha^2} - \eta - 2\kappa\alpha}, \end{aligned}$$

where

$$\begin{aligned} \alpha &\equiv r\phi(\gamma_L + \gamma_M)\sigma_D^2 \\ \eta &\equiv \frac{2r + 3\kappa + 2(r-3\kappa)\alpha}{4}. \end{aligned}$$

The coefficient of the trading strategy Q_ζ is positive if and only if

$$\phi > \underline{\phi} \equiv \frac{\kappa(2r+3\kappa)}{2r(r+\kappa)^2(\gamma_L + \gamma_M)\sigma_D^2}.$$

In this equilibrium, the steady-state stock-holding of the blockholder is

$$\bar{X}_{ss} = \frac{\gamma_M}{\gamma_L + \gamma_M}.$$

It is useful to note that the limit case in Proposition 3 can be interpreted as the case when the shock ζ_t can be fully diversified. The equilibrium in Proposition 3 also corresponds to the limit when $\gamma_L^\epsilon = \epsilon\gamma_L$, $\gamma_M^\epsilon = \epsilon\gamma_M$, and $\sigma_D^\epsilon = \epsilon^{-1/2}\sigma_D$ and ϵ goes to zero. This corresponds to the case in which only the dividend shocks, dB_t^D, are priced while there is no risk premium for the shocks to ζ_t. That is, the shocks to ζ_t are idiosyncratic.[7] For this reason, sometimes we refer to the previous limit as the **limit with idiosyncratic shocks**.

[7]The equilibrium condition depends on σ_D^2 and σ_ζ^2 only through the terms $\gamma_L\sigma_D^2, \gamma_M\sigma_D^2$ and $\gamma_L\sigma_\zeta^2, \gamma_M\sigma_\zeta^2$.

The next proposition examines the limit in which there is a risk premium associated with ζ_t. In this case, the limit of $\bar{X}_{ss}$ differs from that in the observable case. In particular we find that asymmetric information leads to larger blockholder stake in the long run.

Proposition 4. *Consider the small noise limit $\kappa, \sigma_\zeta^2 \to 0$, $\sigma_\zeta^2/2\kappa \to \bar{\sigma}_\zeta^2 > 0$. In the limit, there is a linear Markov perfect Bayesian equilibrium with coefficients*

$$R_q = \frac{\sqrt{(\alpha+1)^2+8\alpha^2}+\alpha+1}{4r^2\phi}$$
$$Q_x = \frac{\sqrt{(\alpha+1)^2+8\alpha^2}-\alpha-1}{4\phi\left(\gamma_L+\gamma_M\right)\sigma_D^2}$$
$$Q_\zeta = \frac{\psi}{2\phi\left(\gamma_L+\gamma_M\right)\sigma_D^2},$$

where $\alpha \equiv r\phi(\gamma_L+\gamma_M)\sigma_D^2$. The steady-state mean stock-holding of the blockholder is

$$\bar{X}_{ss} = \frac{\gamma_M}{\omega_0\gamma_L+\gamma_M},$$

where

$$\omega_0 \equiv 1 - \frac{\left(\gamma_L+\gamma_M\right)\bar{\sigma}_\zeta^2}{\gamma_L\bar{\sigma}_\zeta^2+\frac{1}{2}\left(\gamma_L\bar{\sigma}_\zeta^2+\frac{\phi}{\psi^2}\right)\left(\sqrt{(\alpha+1)^2+8\alpha^2}-\alpha-1\right)} \in \left(-\gamma_M/\gamma_L, 1\right],$$

so $\bar{X}_{ss} > \gamma_M/(\gamma_L+\gamma_M)$.

The market's liquidity R_q^{-1} decreases in risk aversion and the volatility of cash flows σ_D^2 but increases in the cost of effort, ϕ.

Proposition 4 reveals that under asymmetric information the cost of effort ϕ does affect the stationary blockholder stake, contrary to the case under symmetric information. In the asymmetric information case, the more efficient the blockholder (lower ϕ), the less liquid the market (higher R_q) and the larger the stake the blockholder holds in the long-run (higher $\bar{X}_{ss}$). Indeed, Proposition 4 shows that, in steady state, the blockholder's stake is larger than under symmetric information, more so the larger is the volatility of ability shocks (σ_ζ^2). This effect holds as long as there is a risk premium associated with variation in blockholder ability. However, moral hazard is not strictly required: the blockholder holds a large stake, even in the absence of moral hazard, that is even if there is no effort but the blockholder has private information about the cash flow evolution.[8]

The mechanism that leads the blockholder to hold more shares for signaling purposes in settings such as Leland and Pyle (1977) is different from ours. Leland and Pyle (1977) is static, as if

[8]For example, we have verified the result holds when the cash flow follows $dD_t = (\mu+\zeta_t)dt + \sigma dB_t$ where ζ_t is privately observed by the blockholder.

the blockholder had commitment power. Over time, the blockholder would have incentives to continue selling, and their analysis suggests that signaling effects would delay the speed at which the blockholder sells his shares, but it does not speak to the size of the stake the blockholder will hold in the long-run. Moreover, contrary to the static intuition, we show that, under information asymmetry, the blockholder may trade a faster rate in response to changes in his private information despite his price impact.

As previously mentioned, the limit in Proposition 4 is such that long-run uncertainty is positive, even though the individual shocks are very small. This happens because shocks are highly persistent, so $\bar{\sigma}_\zeta^2 > 0$. Long-run asymmetric information explains why $\bar{X}_{ss}$ is higher than in the absence of asymmetric information. A necessary condition for the asymmetric information to affect $\bar{X}_{ss}$ is that the ability shocks ζ_t have an effect on the risk premium required by the market to absorb the residual shares $1 - \bar{X}_{ss}$.

Finally, we look at the effect that asymmetric information has on the stock price.

Corollary 1. *Consider the limit equilibrium in Propositions 3 and 4. Suppose that* $1 > 2r\phi(\gamma_L + \gamma_M)\sigma_D^2$ *so an equilibrium with smooth trading exists in the observable case, and let* P^o *and* P^u *be the coefficients in the observable and unobservable case, respectively. Then,*

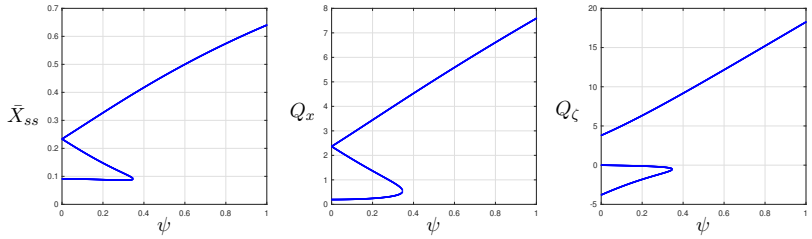
1. *There is* $\kappa^\dagger$ *such that permanent price impact is higher with asymmetric information* $P_x^u > P_x^o$ *if and only if* $\kappa \leq \kappa^\dagger$.

2. *Impact of ability shocks is higher with asymmetric information, that is* $P_\zeta^u > P_\zeta^o$ *if and only if* $\phi \geq \underline{\phi}$ *where* $\underline{\phi}$ *is defined in Proposition 3.*

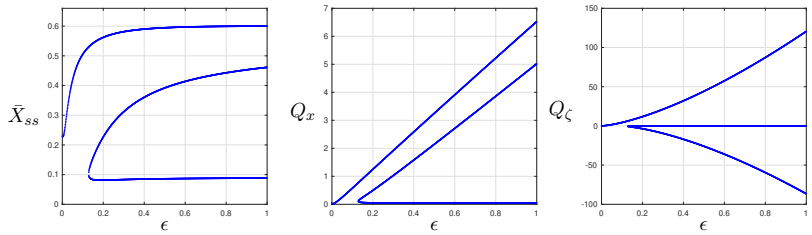
This corollary studies the impact of liquidity, generated by asymmetric information, on stock prices. Previous literature looking at the impact of liquidity on blockholder's intervention has suggested that illiquid markets are beneficial because they encourage blockholder monitoring. This idea is consistent with the intuition following the literature on signaling (Leland and Pyle, 1977). However, Corollary 1 shows that this is only if the case if ability shocks are sufficiently persistent. Later on, in section 5.2.2, we show that if shocks are highly transitory, so private information is short lived, the blockholder trades more aggressively under information asymmetry, which reduces the impact of blockholding on prices. Similarly, ability shocks have a larger impact on prices if the marginal cost of effort is sufficiently high.

5.1 Multiplicity

The feedback between stock prices and firm productivity may lead to multiple equilibria. This result is reminiscent of the feedback effects surveyed by Bond, Edmans, and Goldstein (2012).



(a) Equilibria for different values of ψ. Parameter values: $\gamma_M = 1$, $\gamma_L = 10$, $\sigma_D = 1$, $\sigma_\zeta = 1$, $\kappa = 0.5$, $\phi = 1$, $r = 0.05$, $\mu_D = 1$.



(b) Equilibria when the mean reversion and volatility of shocks are $\epsilon\kappa$ and $\sqrt{\epsilon}\sigma_\zeta$. Parameter values: $\gamma_M = 1$, $\gamma_L = 10$, $\sigma_D = 1$, $\sigma_\zeta = 0.1$, $\kappa = 0.5$, $\phi = 1$, $r = 0.05$, $\mu_D = 1$.

Figure 2: Equilibrium Multiplicity.

If we look at the plot of Q_ζ in Figure 2, we see that for low ψ there are three equilibria, and two of them feature a negative coefficient Q_ζ, which is consistent with the symmetric information case. One of these equilibria yields $Q_\zeta = 0$ as ψ goes to zero, thus converging to the unique Markov equilibrium of the symmetric information case. By contrast, the bottom equilibrium, converges to a strictly negative coefficient Q_ζ, which represents a situation in which trading depends on the blockholder's ability despite ζ_t is payoff irrelevant; hence in the limit, this is not a Markov equilibrium. The upper branch of the correspondence, depicts an equilibrium with positive Q_ζ. This is the only equilibrium that survives when ψ is large, which is the case in which we focus.

Similarly, figure 2(b) shows the equilibrium correspondence for the limit considered in Proposition 4. This figure shows the equilibrium correspondence for $\sigma_\zeta^\epsilon = \sqrt{\epsilon}\sigma_\zeta$ and $\kappa^\epsilon = \epsilon\kappa$ as ϵ goes to zero. For ϵ close to zero, the equilibrium is unique, and coincides with the equilibrium in Proposition 4. However, for larger values of ϵ there are three equilibria, two of them with negative values of Q_ζ and one in which Q_ζ is positive. The latter equilibrium is the one featuring larger ownership by the blockholder in the long-run (i.e. $\bar{X}_{ss}$).

5.2 Dynamics and Steady State

In this section we study the effect of information asymmetry on the dynamics of block-holding, firm productivity and stock prices. As an intermediate step, we analyze the stationary distribution of the two state variables, X_t and ζ_t.

The blockholder's stake is determined by the solution to a linear system of stochastic differential equations for (X_t, ζ_t), and the solution for X_t is given by (see, e.g. Evans (2012)):

$$X_t = \bar{X}_{ss} + e^{-Q_x t}\left(X_0 - \bar{X}_{ss}\right) + \frac{\left(e^{-\kappa t} - e^{-Q_x t}\right) Q_\zeta}{Q_x - \kappa}\zeta_0 + \int_0^t \frac{Q_\zeta\left(e^{-\kappa(t-s)} - e^{-Q_x(t-s)}\right)}{Q_x - \kappa}\sigma_\zeta dB_s^\zeta. \quad (12)$$

From this equation, we arrive at[9]

$$\begin{aligned}
\mathbb{E}[X_t] &= \bar{X}_{ss} + e^{-Q_x t}\left(X_0 - \bar{X}_{ss}\right) + \frac{\left(e^{-\kappa t} - e^{-Q_x t}\right) Q_\zeta}{Q_x - \kappa}\zeta_0 \\
\mathbb{V}[X_t] &= \frac{Q_\zeta^2 \sigma_\zeta^2}{(\kappa - Q_x)^2}\left[\frac{1 - e^{-2Q_x t}}{2Q_x} + \frac{1 - e^{-2\kappa t}}{2\kappa} - \frac{2\left(1 - e^{-(\kappa + Q_x)t}\right)}{\kappa + Q_x}\right] \\
\mathbb{C}\mathrm{ov}[X_t, \zeta_t] &= \frac{Q_\zeta \sigma_\zeta^2}{(\kappa - Q_x)}\left[\frac{\left(1 - e^{-(\kappa + Q_x)t}\right)}{\kappa + Q_x} - \frac{1 - e^{-2\kappa t}}{2\kappa}\right].
\end{aligned}$$

Taking the limit as $t \to \infty$ we find that (X_t, ζ_t) converges to the following stationary distribution

$$N\left[\begin{pmatrix} \bar{X}_{ss} \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{\sigma_\zeta^2 Q_\zeta^2}{2\kappa Q_x(\kappa + Q_x)} & \frac{\sigma_\zeta^2 Q_\zeta}{2\kappa(\kappa + Q_x)} \\ \frac{\sigma_\zeta^2 Q_\zeta}{2\kappa(\kappa + Q_x)} & \frac{\sigma_\zeta^2}{2\kappa} \end{pmatrix}\right]. \quad (13)$$

5.2.1 Steady State

Before characterizing the equilibrium dynamics, we study the stationary distribution of the blockholder's stake. The next proposition provides comparative statics for the stationary distribution of holdings and prices, in the small noise limit in Proposition 4 (i.e. $\lim \sigma_\zeta^2/2\kappa = \bar{\sigma}_\zeta^2 > 0$).

Proposition 5. *In the small noise limit equilibrium characterized in Proposition 4:*

1. *The mean steady-state block $\bar{X}_{ss}$ is increasing in ψ and $\bar{\sigma}_\zeta^2$ and decreasing in ϕ and σ_D^2.*

[9] Remember that $\bar{X}_{ss} \equiv Q_0/Q_x$.

2. *The mean steady-state price,* $\bar{p}_{ss} = P_0 + P_x \bar{X}_{ss}$*, is increasing in* ψ *and* $\bar{\sigma}_\zeta^2$*, and decreasing in* ϕ *and* σ_D^2*.*

3. *The stationary variance of* X_t *is*

$$\mathbb{V}[X_t] = \left(\frac{2\bar{\sigma}_\zeta \psi}{\sqrt{(\alpha+1)^2 + 8\alpha^2} - \alpha - 1} \right)^2,$$

where $\alpha \equiv r\phi(\gamma_L + \gamma_M)\sigma_D^2$*. Hence, the long run variance of* X_t *is:*

   (a) *Increasing in* ψ *and* $\bar{\sigma}_\zeta$*.*

   (b) *Decreasing in* ϕ*,* $\gamma_L + \gamma_M$ *and* σ_D^2*.*

As mentioned above, under information asymmetry, the blockholder's long-run stake is larger than under symmetric information. This leads to stronger monitoring and higher firm productivity, and, ultimately, higher cash flows. On the other hand, cash flows are also more volatile under asymmetric information. Hence, the long-run effect of asymmetric information on the expected stock-price is ambiguous and depends on the blockholder's risk aversion, γ^L. In effect, when the blockholder risk aversion is low, the productivity effect dominates the risk effect, leading to a higher stock price. However, when the blockholder's risk aversion is relatively high, the risk premium effect dominates the productivity effect, thereby leading to a lower stock price.

We had found that, under symmetric information, the blockholder stake is independent of the blockholder monitoring productivity, as captured by his cost of effort ϕ. By contrast, Proposition 5 shows that the intuitive relation between the the blockholder productivity and his holdings is recovered under asymmetric information: in effect, a lower cost of effort leads the blockholder to increase his holdings.

5.2.2 Equilibrium Dynamics

Information asymmetry not only distorts the long-run ownership structure, but it also affects the equilibrium dynamics. Here, we address the following question: how quickly does the blockholder builds his stake (or unwinds it) under information asymmetry *vis-a-vis* symmetric information.

Intuitively, one would think that asymmetric information slows down the blockholder trading, due to price impact considerations. Below we show that this relationship is more subtle: under some conditions, the blockholder trades faster, towards the steady state, under asymmetric information, despite the lower liquidity caused by asymmetric information.

Two cases must be considered depending on whether the private information risk, ζ_t, is priced in equilibrium. First, we study the case when the private information risk is diversifiable. Specifically,

the next proposition studies the dynamics of the small noise limit of Proposition 3, when the private information is diversifiable.

Proposition 6. *Consider the small noise limit equilibrium in Proposition 3 (that is, $\sigma_\zeta^2 \to 0$ and $\kappa > 0$, or the limit with idiosyncratic shocks). Suppose that*

$$\frac{1}{2\phi} > r(\gamma_L + \gamma_M)\sigma_D^2,$$

so an equilibrium with smooth trading exists in the observable case, and consider the case where $\zeta_0 = 0$. Let $\mathbb{E}[X_t^u]$ and $\mathbb{E}[X_t^o]$ be the expected path of ownership in the unobservable and observable case, respectively. Then,

- *There is $\kappa^\dagger$ such that asymmetric information reduces speed of adjustment, that is $Q_x^u < Q_x^o$ if and only if $\kappa < \kappa^\dagger$. If $\phi \leq 1/2$ then $\kappa^\dagger = \infty$ and Q_x^u is always less than Q_x^o.*
- *If $\kappa < \kappa^\dagger$, where $\kappa^\dagger$ is defined in equation* (A.31)*, then:*
  1. *if $X_0 > \bar{X}_{ss}$ then $\mathbb{E}[X_t^u] > \mathbb{E}[X_t^o]$, $\mathbb{E}[a_t^u] > \mathbb{E}[a_t^o]$ for all $t > 0$, and*
  2. *if $X_0 < \bar{X}_{ss}$ then $\mathbb{E}[X_t^u] < \mathbb{E}[X_t^o]$, $\mathbb{E}[a_t^u] < \mathbb{E}[a_t^o]$ for all $t > 0$.*
- *If $\kappa > \kappa^\dagger$, then:*
  1. *if $X_0 > \bar{X}_{ss}$ then $\mathbb{E}[X_t^u] < \mathbb{E}[X_t^o]$, $\mathbb{E}[a_t^u] < \mathbb{E}[a_t^o]$ for all $t > 0$, and*
  2. *if $X_0 < \bar{X}_{ss}$ then $\mathbb{E}[X_t^u] > \mathbb{E}[X_t^o]$, $\mathbb{E}[a_t^u] > \mathbb{E}[a_t^o]$ for all $t > 0$.*

This result demonstrates that the notion that information asymmetry increases the duration of the blockholder stake via lower liquidity is true only if the private information is sufficiently persistent. However, when the private information is rather transitory, the blockholder trades more aggressively (i.e., faster) than under symmetric information, to take advantage of his private information (nonetheless, he is not able to do so because in equilibrium his trading pattern reveals his information).

The previous pattern changes when private information entails a risk-premium, as in the limit studied in Proposition 4. We have shown that, in the long-run, the blockholder's stake is higher under asymmetric information. On the other hand, the blockholder trading is always slower under asymmetric information ($Q_x^u \leq Q_x^o$). In summary, though the blockholder holds a larger stake in the long-run, he takes longer to build it under asymmetric information. Hence, whether information asymmetry boosts monitoring or not, depends on the importance of the long vs short-run effects.

Proposition 7. *Consider the small noise limit equilibrium in Proposition 4 (that is, $\sigma_\zeta^2/2\kappa \to \bar{\sigma}_\zeta^2 > 0$), and suppose that*

$$\frac{1}{2\phi} > r(\gamma_L + \gamma_M)\sigma_D^2,$$

so an equilibrium with smooth trading exists in the observable case, and that $\zeta_0 = 0$ and $X_0 \geq 0$. There is $z^\dagger$ and $x_0^\dagger$ such that

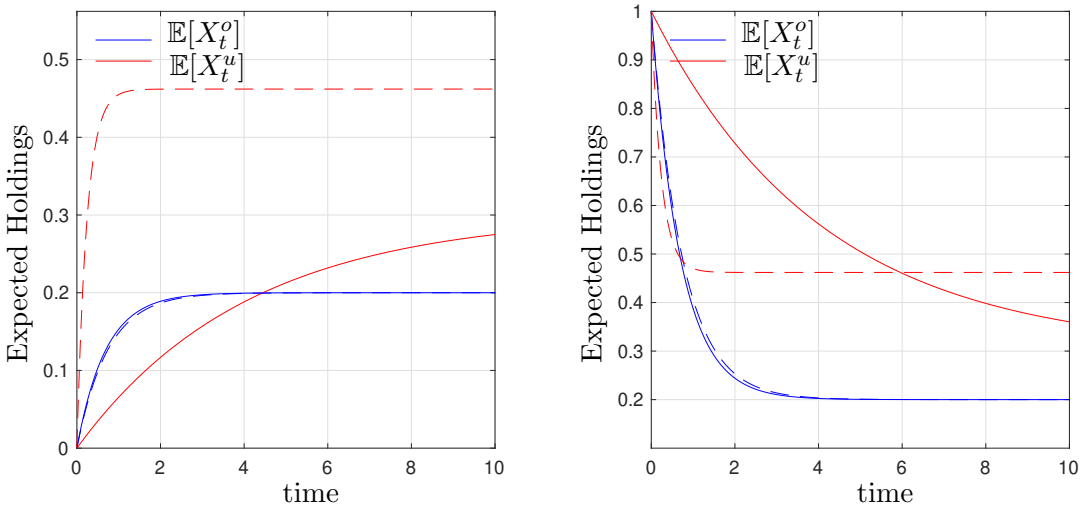
- *If $\psi^2\bar{\sigma}_\zeta^2 < z^\dagger$ and $X_0 < x_0^\dagger$, then there is $0 < t^* < \infty$ such that:*
  1. *For $t < t^*$, the expected blockholding and effort are lower under asymmetric information, that is $\mathbb{E}[X_t^u] < \mathbb{E}[X_t^o]$ and $\mathbb{E}[a_t^u] < \mathbb{E}[a_t^o]$.*
  2. *For $t > t^*$, the expected blockholding and effort are higher with asymmetric information, that is $\mathbb{E}[X_t^u] > \mathbb{E}[X_t^o]$ and $\mathbb{E}[a_t^u] > \mathbb{E}[a_t^o]$.*
- *If $\psi^2\bar{\sigma}_\zeta^2 \geq z^\dagger$ or $X_0 \geq x_0^\dagger$, then, for all $t > 0$, $\mathbb{E}[X_t^u] > \mathbb{E}[X_t^o]$ and $\mathbb{E}[a_t^u] > \mathbb{E}[a_t^o]$.*

In a nutshell, this result indicates that under asymmetric information the expected blockholder's stake single crosses from below (at some point in time) the blockholder's stake under asymmetric information (when $X_0 = 0$). This result speaks to the relationship between liquidity and blockholder monitoring. Some authors have suggested that illiquidity can be beneficial because it reduces the incentives to "cut an run" (Coffee, 1991). The counterargument is that illiquidity is costly because it deters investors from acquiring a large block in the first place (Maug, 1998; Kyle and Vila, 1991; Back et al., 2018). Proposition 7 reconciles these views by showing that illiquidity might indeed reduce the size of the block in the short-term (blocks take longer time to build) but, on the upside, it leads to a larger block in the long-run. The overall impact of information asymmetry depends on which effect dominates, the short- or long-run effect.

6 Liquidity Shocks: Beyond a Fully Revealing Equilibrium

In our baseline model, the blockholder's order flow fully reveals his ability ζ_t. Hence, in equilibrium, there is no asymmetry of information between small investors and the blockholder. Furthermore, conditional on the blockholder's order flow, the firm's cash flow does not provide any additional information to the market. This implies that the blockholder effort choice a_t is myopic: it only depends on the blockholder's stake X_t and his ability ζ_t but not on his reputation $\hat{\zeta}_t$.

In this section, we extend the baseline model and consider a situation in which trading is not fully revealing and, hence, cash flows are informative. Formally, we add a second source of information asymmetry: we assume that the blockholder is subject to unobservable liquidity



(a) Case with $X_0 = 0$ (b) Case with $X_0 = 1$

Figure 3: Expected path of X_t with and without asymmetric information. Parameters: $\gamma_M = 0.5$, $\gamma_L = 2$, $\sigma_D = 1$, $\sigma_\zeta = 0.05$, $\phi = 1.2$, $\psi = 1$, $r = 0.15$, $\mu_D = 1$. In all cases we have taken $\zeta_0 = 0$ (that is, equal to its long-run mean). The solid line corresponds to the case with $\kappa = 0.1$ while the dotted line corresponds to the case with $\kappa = 1$.

shocks, b_t, that reduce his incentive to hold shares. These liquidity shocks are orthogonal to the firm's fundamentals. Thus, similar to Manzano and Vives (2011), Hatchondo, Krusell, and Schneider (2014) and Dávila and Parlatore (2017), we consider a setting in which trading is noisy due to the presence of unobservable liquidity shocks rather than noise trading.

Liquidity shocks affect future trading needs and, due to moral hazard, also affect stock prices in a way that is qualitatively different from noise trading. We depart from traditional models with noise traders for two reasons. First, in practice the blockholder stake is largely observable (albeit with some delay). Second, in our setting, a model with liquidity shocks is more tractable because it requires fewer state variables to characterize the equilibrium.[10]

We assume that the liquidity shocks are privately observed by the blockholder and follow the following Ornstein-Uhlenbeck process:

$$db_t = -\lambda b_t dt + \sigma_b dB_t^b,$$

where λ captures the persistence of liquidity shocks. In turn, the blockholder's wealth process is

[10]Because competitive investors (that is market makers) are risk averse, we cannot model noisy supply as the increments of a Brownian motion as in the traditional Kyle model. In continuous time, this implies that noise traders cannot be i.i.d, which means that, in addition to keep track of X_t, ζ_t and $\hat{\zeta}_t$, we also need to keep track of $\hat{X}_t$, the current noisy supply, and the market beliefs about the current noisy supply

given by

$$dW_t = (rW_t - c_t - R_t(q_t)q_t - \Phi(a_t, \zeta_t) + (\mu_D + a_t - \delta b_t)X_t)dt + X_t\sigma_D dB_t^D.$$

The parameter δ captures the exposure of the blockholder to the liquidity shock. This specification nests the baseline model when $\delta = 0$.

Denoting the market beliefs by $\hat{b}_t = \mathbb{E}[b_t|(D_s, q_s^L)_{s\leq t}]$, $\hat{\zeta}_t = \mathbb{E}[\zeta_t|(D_s, q_s^L)_{s\leq t}]$, then a linear Markov equilibrium is given by an affine function of five variables $(X_t, \zeta_t, b_t, \hat{\zeta}_t, \hat{b}_t)$. As will become clear later, due to the persistence of the liquidity shock b_t, one needs to consider the impact of deviations from the equilibrium trading rate q_t^L, which we denote by Δ_t.

In the sequel, we consider a linear equilibrium, which is characterized by the following strategies:

$$q_t^L = Q_0 - Q_x X_t + Q_\zeta \zeta_t + Q_b b_t + Q_{\hat{b}}\hat{b}_t \tag{14a}$$

$$a_t = A_0 + A_x X_t + A_\zeta \zeta_t + A_b b_t + A_{\hat{b}}\hat{b}_t \tag{14b}$$

$$p_t = P_0 + P_x X_t + P_\zeta \hat{\zeta}_t + P_{\hat{b}}\hat{b}_t \tag{14c}$$

$$R(X_t, \hat{b}_t, q_t) = R_0 + R_x X_t + R_{\hat{b}}\hat{b}_t + R_q q_t \tag{14d}$$

Notice that our conjectured equilibrium strategies q_t^L and a_t depend on $\hat{b}_t$ alone but not $\hat{\zeta}_t$. This holds without loss because, as we show below, the belief $\hat{\zeta}_t$ is uniquely determined by q_t^L and $\hat{b}_t$.

To pin down the equilibrium, we take the following steps. First, we derive the market's beliefs given the conjectured equilibrium by solving for the market's filtering problem. Next we solve the small investor portfolio problem and derive the residual supply faced by the blockholder. Finally, we solve the blockholder's optimization problem.

6.1 Learning

Because the market perfectly observes the order flow q_t^L, the variable

$$I_t \equiv \frac{q_t^L - Q_0 + Q_x X_t - Q_{\hat{b}}\hat{b}_t}{Q_\zeta} = \zeta_t + \frac{Q_b}{Q_\zeta} b_t, \tag{15}$$

is informationally equivalent to the blockholder's order flow q_t^L. From the market perspective, the order flow is thus a noisy signal of ability ζ_t because it is also affected by liquidity shocks b_t. Hence, the market cannot perfectly disentangle the two drivers of blockholder trading, ability and liquidity needs.

The market's filtering problem is non-standard. Unlike in standard Kalman filtering problems, the market observes a linear combination of ζ_t and b_t without any noise, which means that the

covariance matrix of the conditional distribution of (ζ_t, b_t) is singular, so we cannot use standard filtering techniques. Technically, this corresponds to a singular filtering problem (Xiong, 2008).[11]

The key to solving this filtering problem is to transform the original two-dimensional filtering problem for (ζ_t, b_t) into a single dimensional filtering problem for b_t. Then, once we have determined the belief $\hat{b}_t$, we solve for $\hat{\zeta}_t$ using equation (15), specifically, given I_t and $\hat{b}_t$ we have that

$$\hat{\zeta}_t = I_t - \frac{Q_b}{Q_\zeta}\hat{b}_t.$$

On some level, this problem is similar to how the market forms belief $\hat{\zeta}_t$ when the order flow is fully revealing, except that the intercept of the residual supply is time varying and determined by $\hat{b}_t$.

If we differentiate I_t, and use equation (15) to eliminate ζ_t, we get the following SDE for I_t

$$dI_t = -\left(\kappa I_t + (\lambda - \kappa)\frac{Q_b}{Q_\zeta}b_t\right)dt + \sigma_\zeta dB_t^\zeta + \frac{Q_b}{Q_\zeta}\sigma_b dB_t^b. \tag{16}$$

Similarly, substituting the conjectured equilibrium effort, and using equation (15) to substitute ζ_t, we find that the dividend process follows

$$dD_t = \left(\mu_D + A_0 + A_x X_t + A_\zeta I_t - A_\zeta \frac{Q_b}{Q_\zeta}b_t + A_b b_t + A_{\hat{b}}\hat{b}_t\right)dt + \sigma_D dB_t^D. \tag{17}$$

The key step in this derivation is to use (15) to eliminate ζ_t from equations (16) and (17). This allows us to transform our original singular filtering problem for (ζ_t, b_t) into a standard filtering problem for b_t alone in which the information consists of two signals D_t and I_t.

Now, we can use the Kalman-Bucy formula to get the market's belief updating

$$d\hat{b}_t = -\lambda\hat{b}_t dt + \beta_q\left(\sigma_\zeta d\tilde{B}_t^\zeta + \frac{Q_b}{Q_\zeta}\sigma_b d\tilde{B}_t^b\right) + \beta_D\sigma_D d\tilde{B}_t^D, \tag{18}$$

where $(\tilde{B}_t^\zeta, \tilde{B}_t^b, \tilde{B}_t^D)$ are Brownian motions under the filtration generated by $(q_t, D_t)_{t\geq 0}$. In a stationary linear equilibrium, the covariance matrix of $(\hat{b}_t, \hat{\zeta}_t)$ is constant.

Because we only need to keep track of $\hat{b}_t$, this amounts to looking for the stationary solution of the differential equation for the conditional variance of b_t, which we denote by $\sigma_{\hat{b}}^2 \equiv \mathbb{V}[b_t|\mathcal{F}_t^{q,D}]$. Given equation (18), we can use equations (15) and (16) to derive a stochastic differential equation for $\hat{\zeta}_t$. In the appendix, we show that the evolution of the vector $(\hat{\zeta}_t, \hat{b}_t)$ is given by the following

[11] More generally, this is a filtering problem with Ornstein-Uhlenbeck noise. The theory of filtering for general Gaussian process is developed in Kunita (1993). The specific case with Ornstein-Uhlenbeck noise is developed in detail in Liu and Xiong (2010).

lemma.

Lemma 3. *$\hat{\zeta}_t$ and $\hat{b}_t$ satisfy the following stochastic differential equations*

$$d\hat{b}_t = -\lambda \hat{b}_t dt + \beta_q \left(\sigma_\zeta d\tilde{B}_t^\zeta + \frac{Q_b}{Q_\zeta} \sigma_b d\tilde{B}_t^b \right) + \beta_D \sigma_D d\tilde{B}_t^D \tag{19a}$$

$$d\hat{\zeta}_t = -\kappa \hat{\zeta}_t dt + \left(1 - \beta_q \frac{Q_b}{Q_\zeta} \right) \left(\sigma_\zeta d\tilde{B}_t^\zeta + \sigma_b \frac{Q_b}{Q_\zeta} d\tilde{B}_t^b \right) - \beta_D \frac{Q_b}{Q_\zeta} \sigma_D d\tilde{B}_t^D, \tag{19b}$$

$$\begin{aligned} d\tilde{B}_t^\zeta &= \sigma_\zeta^{-1} (d\zeta_t + \kappa \hat{\zeta}_t dt) \\ d\tilde{B}_t^b &= \sigma_b^{-1} (db_t + \lambda \hat{b} dt) \\ d\tilde{B}_t^D &= \sigma_D^{-1} (dD_t - (\mu_D + E_t(a_t)) dt) \end{aligned}$$

where

$$\begin{aligned} \beta_q &= \frac{\sigma_b^2 + (\kappa - \lambda)\sigma_{\hat{b}}^2}{\left(\frac{Q_b}{Q_\zeta}\right)^2 \sigma_b^2 + \sigma_\zeta^2} \frac{Q_b}{Q_\zeta} \\ \beta_D &= \frac{\sigma_{\hat{b}}^2}{\sigma_D^2} \left(A_b - A_\zeta \frac{Q_b}{Q_\zeta} \right) \end{aligned}$$

and

$$0 = -2\lambda\sigma_{\hat{b}}^2 + \sigma_b^2 - \left[\frac{\left(\sigma_b^2 + (\kappa - \lambda)\sigma_{\hat{b}}^2\right)^2}{\sigma_\zeta^2 + \left(\frac{Q_b}{Q_\zeta}\right)^2 \sigma_b^2} \left(\frac{Q_b}{Q_\zeta} \right)^2 + \frac{\sigma_{\hat{b}}^4}{\sigma_D^2} \left(A_b - A_\zeta \frac{Q_b}{Q_\zeta} \right)^2 \right]. \tag{20}$$

The innovation processes $(\tilde{B}_t^\zeta, \tilde{B}_t^b, \tilde{B}_t^D)$ are standard Brownian motions with respect to the filtration $(\mathcal{F}_t^M)_{t \geq 0}$.

The sensitivity of beliefs to order-flow surprises or dividend surprises depend on how blockholder trading and effort react to liquidity and ability shocks, and the speed of mean reversion of these variables. For example, if the blockholder's order flow is increasing in both b_t and ζ_t (Q_ζ and Q_b are positive), then market beliefs about liquidity shock ($\hat{b}_t$) increase after positive trading surprises. This means that the market attributes part of the increase in blockholder stake to liquidity shocks.

The impact of unexpected trading on reputation $\hat{\zeta}_t$ depends on how sensitive is trading to ability shocks —relative to liquidity shocks. The reaction of market beliefs to unexpected dividend shocks depends on the magnitude of A_b/A_ζ relative to Q_b/Q_ζ.

The last step before analyzing the blockholder's optimization problem is to analyze the evolution of market beliefs $\hat{b}_t$ given the blockholder's information set and arbitrary effort and trading strategies

$(\tilde{a}_t, \tilde{q}_t)$, which might differ from the equilibrium conjecture in (14a) and (14b). In other words, we study how deviations from the equilibrium affect market beliefs.

Lemma 4. *Suppose the market believes that the blockholder strategy is given by equation* (14a) *and* (14b) *but the blockholder follows the strategy* $(\tilde{q}_t, \tilde{a}_t)$*, where* $\tilde{q}_t = q_t + \Delta_t$*. Given the blochkolder's information, the market belief* $\hat{b}_t$ *follows the following stochastic differential equation*

$$d\hat{b}_t \;=\; \Big(\mu_{\hat{b}}(X_t, b_t, \hat{b}_t, \zeta_t, \Delta_t) \;+\; \beta_D \tilde{a}_t\Big)dt \;+\; \frac{\beta_q}{Q_\zeta} d\Delta_t \;+\; \beta_q \sigma_\zeta dB_t^\zeta \;+\; \beta_q \frac{Q_b}{Q_\zeta}\sigma_b dB_t^b \;+\; \beta_D \sigma_D dB_t^D$$

where

$$\mu_{\hat{b}}(X_t, b_t, \hat{b}_t, \zeta_t, \Delta_t) = \mathcal{B}_0 + \mathcal{B}_x X_t + \mathcal{B}_b b_t - \mathcal{B}_{\hat{b}} \hat{b}_t + \mathcal{B}_\zeta \zeta_t + \mathcal{B}_\Delta \Delta_t$$

and $(\mathcal{B}_0, \mathcal{B}_x, \mathcal{B}_b, \mathcal{B}_{\hat{b}}, \mathcal{B}_\zeta, \mathcal{B}_\Delta)$ *are coefficients provided in* (B.4)-(B.6)*.*

6.2 Optimal Strategy and Equilibrium

Given the characterization of small investor beliefs in Proposition 3, we can pin down their portfolio optimization. The small investors solve the following stochastic control problem

$$\max_{c, q^M} \mathbb{E}_t^M \left[\int_t^\infty e^{-r(s-t)} u_M(c_s) ds\right]$$

subject to

$$\begin{aligned}
dW_t &= (rW_t - c_t - p_t q_t^M + (\mu_D + A_0 + A_x X_t + A_\zeta \hat{\zeta}_t + (A_b + A_{\hat{b}})\hat{b}_t) Y_t)dt + \sigma_D Y_t d\tilde{B}_t^D \\
dY_t &= q_t^M dt \\
dX_t &= \left(Q_0 - Q_x X_t + Q_\zeta \hat{\zeta}_t + \left(Q_b + Q_{\hat{b}}\right)\hat{b}_t\right) dt.
\end{aligned}$$

Because investors do not observe b_t, the coefficients of $\hat{b}_t$ in the law of motion of D_t and X_t given their information set, are the sum of the coefficients of b_t and $\hat{b}_t$ in the blockholder's strategy.

As in the baseline model, we conjecture a value function of the form

$$J(W, Y, X, \hat{b}, \hat{\zeta}) = -\frac{\exp\left(-r\gamma_M \left(W_M + H(Y, X, \hat{b}, \hat{\zeta})\right)\right)}{r},$$

and show that the certainty equivalent H satisfies an HJB equation analogous to the one in (1). In particular, we have the following Lemma.

Proposition 8. *The certainty equivalent H satisfies the HJB equation*

$$\begin{aligned} rH = \max_q \; (\mu_D + A_0 + A_x X + A_\zeta \hat{\zeta} + (A_b + A_{\hat{b}})\hat{b})Y - p(X,\hat{\zeta},\hat{b})q - \frac{1}{2} r\gamma_M \left[\sigma_D^2 Y^2 + 2\beta_D \sigma_D^2 \left(H_{\hat{b}} - \frac{Q_b}{Q_\zeta} H_{\hat{\zeta}}\right) Y \right. \\ \left. + \Sigma_{\hat{b}}^2 H_{\hat{b}}^2 + \Sigma_{\hat{\zeta}}^2 H_{\hat{\zeta}}^2 + 2\Sigma_{\hat{b}\hat{\zeta}} H_{\hat{b}} H_{\hat{\zeta}}\right] + \left(Q_0 - Q_x X + Q_\zeta \hat{\zeta} + \left(Q_b + Q_{\hat{b}}\right)\hat{b}\right) H_x \\ + q H_y - \kappa \hat{\zeta} H_{\hat{\zeta}} - \lambda \hat{b} H_{\hat{b}} + \frac{1}{2}\left[\Sigma_{\hat{b}}^2 H_{\hat{b}\hat{b}} + \Sigma_{\hat{\zeta}}^2 H_{\hat{\zeta}\hat{\zeta}} + 2\Sigma_{\hat{b}\hat{\zeta}} H_{\hat{b}\hat{\zeta}}\right], \end{aligned}$$

where the coefficients $(\Sigma_{\hat{b}}^2, \Sigma_{\hat{\zeta}}^2, \Sigma_{\hat{b}\hat{\zeta}})$ correspond to the quadratic variation and covariation of $\hat{b}_t$ and $\hat{\zeta}_t$, respectively, which are provided in (B.7)-(B.9).

We guess and verify that the certainty equivalent is given by a quadratic function of the form

$$H(Y, X, \hat{b}, \hat{\zeta}) = h_0 + h_y Y + h_{\hat{\zeta}}\hat{\zeta} + h_{\hat{b}}\hat{b} + h_{xy} XY + h_{y\hat{\zeta}}\hat{\zeta} Y + h_{y\hat{b}}\hat{b} Y + h_{yy} Y^2,$$

where the coefficients are provided in equations (B.15a)-(B.15d) in the appendix. As before, taking the first order condition from the HJB equation, and invoking the market clearing condition $X_t + Y_t = 1$, yields the equilibrium price as given by

$$\begin{aligned} p_t &= H_y(Y_t, X_t, \hat{b}_t, \hat{\zeta}_t)\Big|_{Y_t = 1 - X_t} \\ &= P_0 + P_x X_t + P_\zeta \hat{\zeta}_t + P_{\hat{b}} \hat{b}_t. \end{aligned} \tag{21}$$

As in the case without liquidity shocks, we can derive the residual demand combining the price function in (14c) with equation the equation for $\hat{\zeta}_t$ in equation (15), which yields

$$\begin{aligned} R(X_t, \hat{b}_t, q_t) &= P_0 + P_x X_t + \frac{P_{\hat{\zeta}}}{Q_\zeta}\left(q_t - Q_0 + Q_x X_t - (Q_{\hat{b}} + Q_b)\hat{b}_t\right) + P_{\hat{b}}\hat{b}_t \\ &= P_0 - P_{\hat{\zeta}}\frac{Q_0}{Q_\zeta} + \left(P_x + P_{\hat{\zeta}}\frac{Q_x}{Q_\zeta}\right) X_t + \left(P_{\hat{b}} - \frac{Q_b + Q_{\hat{b}}}{Q_\zeta} P_{\hat{\zeta}}\right)\hat{b}_t + \frac{P_{\hat{\zeta}}}{Q_\zeta} q_t \\ &= R_0 + R_x X_t + R_{\hat{b}}\hat{b}_t + R_q q_t. \end{aligned}$$

Next, we can formulate the blockholder problem. Because shocks are mean reverting, we need to consider deviations in the rate of change of the order flow q_t. Hence, if we consider a trading

strategy $\tilde{q}_t = q_t + \Delta_t$, where $d\Delta_t = \dot{\Delta}_t dt$, we can write the problem of the blockholder as follows

$$\max_{(c_t,\dot{\Delta}_t,a_t)_{t\geq 0}} \mathbb{E}_0^L \left[\int_0^\infty e^{-rt} u_L(c_t) dt\right]$$

subject to

$$dW_t = (rW_t - c_t - \Phi(a_t,\zeta_t) - R(X_t,\hat{b}_t,q_t^L+\Delta)(q_t^L+\Delta) + (\mu_D + a_t - \delta b_t)X_t)dt + X_t\sigma_D dB_t^D$$

$$d\hat{b}_t = \left(\mu_{\hat{b}}(X_t,b_t,\hat{b}_t,\zeta_t,\Delta_t) + \frac{\beta_q}{Q_\zeta}\dot{\Delta}_t + \beta_D a_t\right)dt + \beta_q\sigma_\zeta dB_t^\zeta + \beta_q\frac{Q_b}{Q_\zeta}\sigma_b dB_t^b + \beta_D\sigma_D dB_t^D$$

$$dX_t = (q_t^L + \Delta_t)dt$$

$$d\Delta = \dot{\Delta}_t dt.$$

We guess and verify that the value function again takes the exponential form. The certainty equivalent G satisfies an HJB equation similar to that in the case without liquidity shocks. Notice that $\Delta_t = 0$ on equilibrium path, so the certainty equivalent is $G(X,\zeta,b,\hat{b},0)$. The verification argument used here differs from the standard one in stochastic control. We construct a verification function $V(W,X,\zeta,b,\hat{b},\Delta)$ which only corresponds to the value function on-the-equilibrium path. Off-the-equilibrium path, $V(W,X,\zeta,b,\hat{b},\Delta)$ provides an upper bound to the continuation payoff that the blockholder can get from a deviation, which allows us to verify the optimality of our conjectured optimal strategy using $V(W,X,\zeta,b,\hat{b},\Delta)$.

Proposition 9. *Let*

$$V(W,X,\zeta,b,\hat{b},\Delta) = -\frac{\exp\left(-r\gamma_L\left(W + G(X,\zeta,b,\hat{b},\Delta)\right)\right)}{r},$$

where G satisfies the HJB equation

$$\begin{aligned} rG = \max_a\ & (\mu_D + a - \delta b)X - R(X,\hat{b},q^L+\Delta)(q^L+\Delta) - \Phi(a,\zeta) \\ & - \frac{r\gamma_L}{2}\left[\sigma_D^2X^2 + 2\sigma_D^2\beta_D G_{\hat{b}}X + \Sigma_{\hat{b}}G_{\hat{b}}^2 + \sigma_b^2G_b^2 + \sigma_\zeta G_\zeta^2 + \beta_q\sigma_\zeta^2 G_{\hat{b}}G_\zeta + \beta_q\frac{Q_b}{Q_\zeta}\sigma_b^2 G_{\hat{b}}G_b\right] \\ & - \kappa\zeta G_\zeta - \lambda b G_b + \left(\mu_{\hat{b}}(X,b,\hat{b},\zeta,\Delta) + \beta_D a\right)G_{\hat{b}} + (q^L+\Delta)G_X \\ & + \frac{1}{2}\left[\Sigma_{\hat{b}}G_{\hat{b}\hat{b}} + \sigma_b^2G_{bb} + \sigma_\zeta^2G_{\zeta\zeta} + \beta_q\sigma_\zeta^2G_{\hat{b}\zeta} + \beta_q\frac{Q_b}{Q_\zeta}\sigma_b^2G_{\hat{b}b}\right]. \end{aligned}$$

If the following optimality conditions are satisfied for all $(X_t, \zeta_t, b_t, \hat{b}_t, \Delta_t)$

$$G_\Delta(X_t, \zeta_t, b_t, \hat{b}_t, 0) + \frac{\beta_q}{Q_\zeta} G_{\hat{b}}(X_t, \zeta_t, b_t, \hat{b}_t, 0) = 0 \tag{22a}$$

$$\Big(G_\Delta(X_t, \zeta_t, b_t, \hat{b}_t, \Delta_t) + \frac{\beta_q}{Q_\zeta} G_{\hat{b}}(X_t, \zeta_t, b_t, \hat{b}_t, \Delta_t)\Big)\Delta_t \leq 0, \tag{22b}$$

then the trading strategy q_t^L *in equation* (14a) *is incentive compatible and, on the equilibrium path, the blockholder continuation value is* $V(W_t, X_t, \zeta_t, b_t, \hat{b}_t, 0)$.

It can be verified that the certainty equivalent G is given by a linear quadratic function of the form

$$\begin{aligned} G(X, \zeta, b, \hat{b}, \Delta) = g_0 + g_x X + g_\zeta \zeta + g_b b + g_{\hat{b}} \hat{b} + g_\Delta \Delta + g_{x\zeta} \zeta X + g_{xb} bX + g_{x\hat{b}} \hat{b} X \\ + g_{\zeta b} \zeta b + g_{\Delta x} \Delta X + g_{\Delta\zeta} \Delta\zeta + g_{\Delta b} \Delta b + g_{\Delta\hat{b}} \Delta\hat{b} + g_{\zeta\hat{b}} \zeta\hat{b} + g_{b\hat{b}} b\hat{b} + g_{xx} X^2 \\ + g_{\zeta\zeta} \zeta^2 + g_{bb} b^2 + g_{\hat{b}\hat{b}} \hat{b}^2 + g_{\Delta\Delta} \Delta^2. \end{aligned}$$

If we combine equations (22a) and (22b), we get that (22b) is satisfied only if the coefficients satisfy the following inequality

$$g_{\Delta\Delta} + \frac{\beta_q}{Q_\zeta} g_{b\hat{b}} \leq 0. \tag{23}$$

The system of equations satisfied by the coefficients can be found in Section B.1 in the appendix. The proof of Proposition 9 requires to address the fact that the function V corresponds to the value function only on-the-equilibrium path, and consider global deviations rather than only local ones. Equation (22a) is a local incentive compatibility constraint so that $\dot{\Delta}_t = 0$ is optimal on the equilibrium path when $\Delta_t = 0$. However, the fact that the blockholder cannot benefit from a local deviation does not imply he cannot benefit from a global one. The function V is constructed under the assumption that following any deviation with $\Delta_t = \tilde{q}_t - q_t$, the blockholder follows the trading strategy $\tilde{q}_s = q_s + \Delta_t$, $s > t$. That is, the blockholder keeps adjusting the order flow at the same rate as before the deviation, which means that the deviation is permanent. In the verification argument we show that, if such a deviation is suboptimal, then any global deviation is also suboptimal.

Finally, we need to verify that the vector $(\zeta_t, b_t, \hat{b}_t, X_t)$ converges to a stationary distribution (that is, that the linear system of SDEs describing the evolution of $(\zeta_t, b_t, \hat{b}_t, X_t)$ is stable), which

amounts to verifying that $Q_x > 0$.[12] Taking the first order condition in the HJB equation we get that on-the-equilibrium-path the effort strategy is given by

$$a_t = \frac{\psi\zeta_t + X_t + \beta_D G_{\hat{b}}(X_t, \zeta_t, b_t, \hat{b}_t, 0)}{2\phi}. \tag{24}$$

The solution to the blockholder strategy in (24) is difficult to interpret. However, we can obtain some intuition about the effect of reputation using the following representation for the equilibrium strategies

Proposition 10. *The equilibrium effort a_t satisfies*

$$a_t = \frac{\psi\zeta_t + X_t}{2\phi} + \frac{\beta_D}{2\phi}\,\mathbb{E}_t^L\left[\int_t^\infty e^{-(r+\mathcal{B}_{\hat{b}})(s-t)}\,\frac{u_L'(c_s^L)}{u_L'(c_t^L)}R_{\hat{b}}q_s^L ds\right]. \tag{25}$$

Equation (25) reveals a fundamental difference between the baseline model and the model with liquidity shocks. The first term corresponds to the optimal effort in the fully revealing equilibrium, while the second term captures the impact of reputation concerns. Under the baseline model, effort is myopic because cash flows do not provide incremental information about ability, relative to the order flow. By contrast, with liquidity shocks, effort is forward looking. Effort has long-term implications because, by altering the cash flow, the blockholder's effort affects the market belief about ability, hence the price the blockholder will pay on future trades. The incentive to exert or cut effort is determined by the impact of beliefs in the future residual supply faced by the blockholder, weighted by the blockholder's stochastic discount factor. This effect is discounted at $\mathcal{B}_{\hat{b}}$, which captures the mean reversion of beliefs under the blockholder's information set $\mathcal{F}_t^L$.

In this context, a positive shock may induce the blockholder to reduce his effort to depress the cash flows and thus draw the market belief down. The blockholder has an incentive to depress cash flows so the market interprets his buying new shares as driven by liquidity needs rather than higher ability. This effect can be seen by looking at equations (25).

[12] We need to verify that all eigenvalues of

$$\begin{pmatrix} -\kappa & 0 & 0 & 0 \\ 0 & -\lambda & 0 & 0 \\ Q_\zeta & Q_b & -Q_x & Q_{\hat{b}} \\ \mathcal{B}_\zeta + \beta_D A_\zeta & \mathcal{B}_b + \beta_D A_b & \mathcal{B}_x + \beta_D A_x & -\mathcal{B}_{\hat{b}} + \beta_D A_{\hat{b}} \end{pmatrix}$$

are negative. However, by the properties of the determinant of a block matrix, we only need to check the eigenvalues of the lower block which are $-Q_x$ and $-\mathcal{B}_{\hat{b}} + \beta_D A_{\hat{b}}$ as $\mathcal{B}_x + \beta_D A_x = 0$. Substituting $\mathcal{B}_{\hat{b}}$ we find that $-\mathcal{B}_{\hat{b}} + \beta_D A_{\hat{b}} = -\lambda - \frac{\sigma_D^2}{\sigma_{\hat{b}}^2}\beta_D^2 < 0$, so we only need to verify that $Q_x > 0$.

6.3 Numerical Example

To obtain the equilibrium one needs to solve a large system of polynomial equations. Because it is not possible to solve this system in closed form, we look at two numerical examples that illustrate the interaction between the blockholder's incentives to work and trade, and highlight the mechanism behind the dynamics.

Table 1 presents the coefficients of the equilibrium for several values of the mean reversion of liquidity shocks λ. In particular, we focus in two examples. In the first example, the liquidity shock is more persistent than the ability shock ($\lambda = 0.1$) while in the second example the liquidity shock is relatively transitory ($\lambda = 0.5$). Notice that when the blockholder stake has an impact on ability, liquidity shocks affect the firm fundamentals. A high liquidity shock suggests that the blockholder is likely to sell in the future, which anticipates a reduction in productivity. If liquidity shocks are more persistent, they are also more relevant for valuation, as they have long lasting effects on productivity. However, when liquidity shocks are transitory their main role is to obscure the trading motives of the blockholder.

Steady State Table 1 shows the expected blockholder stake and effort as well as the price under the stationary distribution for both the case with and without liquidity shocks.

If liquidity shocks are less persistent than ability shocks, then the presence of liquidity shocks reduces the blockholder stake, effort, and ultimately leads to a lower stock price. The effect of liquidity shocks is milder as the shocks become more transitory (that is, a higher value of λ). Efficient risk sharing requires that the blockholder holds a smaller block in this case. If shocks are less persistent, the long-run variance of liquidity shocks is reduced and so its impact on risk-sharing. The situation is qualitatively different when liquidity shocks are more persistent than ability shocks: then the average blockholder stake can even be higher with liquidity shocks than in the benchmark. In that case, liquidity shocks may decrease liquidity and exacerbate the ownership concentration.

Effort and Trading Strategy The coefficients of the effort strategy a_t in table 1, capture the Ratchet effect identified in Proposition 10. Specifically, the negative intercept and lower coefficients on ability, in the presence of liquidity shocks, capture the blockholder's tendency to distort effort due to the Ratchet effect. A positive ability shock leads the blockholder to buy shares, and this generates incentives to reduce effort to depress cash flows and lower the price of the shares he intends to buy.

The effect of block size on effort is apparent when we look at the level of current effort relative

Equilibrium Strategy

λ	0.1	0.5	1.0	1.5	Benchmark
			Effort		
A_0	-1.23	-0.19	0.00	0.00	0.00
A_x	1.91	1.28	1.00	1.00	1.00
A_ζ	-7.36	-0.23	0.99	1.00	1.00
A_b	5.67	-0.62	0.00	0.00	
$A_{\hat{b}}$	-3.22	0.48	0.00	0.00	
			Trading		
Q_0	8.05	2.05	2.63	2.69	2.74
Q_x	7.31	3.08	3.47	3.51	3.53
Q_ζ	52.40	12.44	14.54	14.80	14.98
Q_b	-36.22	5.83	0.94	0.32	
$Q_{\hat{b}}$	20.56	-4.44	-0.50	-0.12	
			Residual Supply		
R_0	13.66	14.02	13.37	13.34	13.32
R_x	23.32	20.90	23.06	23.23	23.34
$R_{\hat{b}}$	-3.33	-1.40	-0.51	-0.24	
R_q	0.47	1.56	1.41	1.40	1.38
			Beliefs		
β_D	0.05	-0.02	0.00	0.00	
β_q	-0.57	0.28	0.04	0.01	
$1-\beta_q\frac{Q_b}{Q_\zeta}$	0.60	0.87	1.00	1.00	
$-\beta_D\frac{Q_b}{Q_\zeta}$	0.04	0.01	0.00	0.00	
$\sigma^2_{\hat{b}}$	0.09	0.04	0.02	0.01	
			Steady State		
$\bar{X}_{ss}$	1.10	0.67	0.76	0.77	0.78
$\bar{p}_{ss}$	39.34	27.92	30.87	31.19	31.41
$\bar{a}_{ss}$	0.87	0.66	0.76	0.77	0.78

Table 1: Coefficients Equilibrium. Parameters: Parameters: $\gamma_M = 1.0$, $\gamma_L = 10.0$, $\sigma_D = 1.0$, $\sigma_\zeta = 0.2$, $\sigma_b = 0.2$, $\kappa = 0.2$, $\phi = 0.5$, $\psi = 1.0$, $r = 0.05$, $\mu_D = 1.0$.

to the steady state, which is given by

$$a_t - \bar{a}_{ss} = A_x(X_t - \bar{X}_{ss}) + A_\zeta \zeta_t + A_b b_t + A_{\hat{b}}\hat{b}_t \tag{26}$$

The coefficient A_x is higher than in the benchmark because the blockholder has an incentive to over supply effort if his stake is above its long-term target so he expects to sell shares. This effect

is more pronounced when liquidity shocks are persistent as in this case the blockholder adjusts his portfolio more aggressively – this is captured by the higher coefficient Q_x.

Finally, if we look at the impact of liquidity shocks on effort, we notice that this effect is driven by the blockholder's trading strategy. The blockholder sells shares in response to liquidity shocks only if these shocks are sufficiently persistent, in which case he increases effort to increase the selling price. However, if liquidity shocks are transitory, the blockholder actually buys shares in response to liquidity shocks, and at the same time reduces his effort to lower the price through lower cash flows. It is counterintuitive that the blockholder buys shares when holding them is more costly for him. The benefit of doing is only understood once we consider the dynamics of trading that follow the liquidity shock, as captured by the impulse response function in Figure 4 which we discuss next.

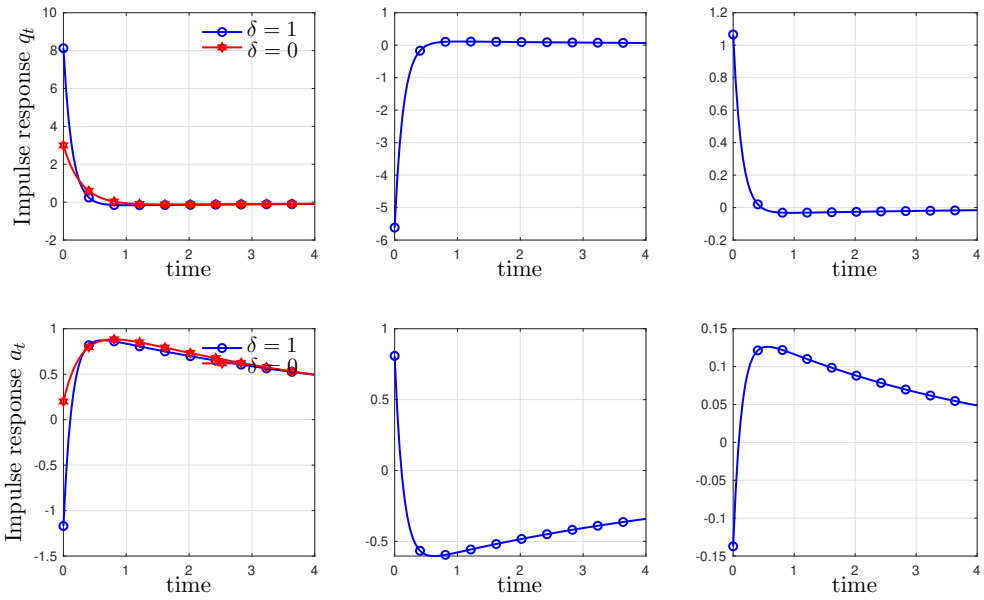
Impulse Response Function Next, we discuss how ability, liquidity, and dividend shocks, affect the dynamics. We again distinguish two cases, the case in which liquidity shocks are more persistent than ability shocks, and vice-versa. As previously mentioned, when the blockholder can influence the value of the firm, liquidity shocks affect the firm's fundamentals. If these liquidity shocks are more persistent than ability shocks, then valuation is driven more by beliefs about liquidity shocks than ability shocks. On the other hand, if liquidity shocks are transitory, then beliefs about ability are more relevant.

We provide the case when liquidity shocks are transitory ($\lambda = 0.5$) in figure 4 and the case in which liquidity shocks are permanent ($\lambda = 0.1$) in figure 5 . Figures 4(a) and 5(a) show the impulse response for ability shocks (ζ_t). As in the baseline, a positive ability shock leads the blockholder to buy shares. Whether liquidity shocks increases the trading rate depends on the persistence of liquidity shocks. However, regardless of persistence, the Ratchet effect leads to a reduction in effort, in contrast to the baseline where increments in ability always increase effort.

Figures 4(b) and 5(b) show the impact of liquidity shocks. A liquidity shocks leads the blockholder to sell shares when liquidity shocks are persistent, and this lead the blockholder to increases effort to boost the selling price of his shares. However, if liquidity shocks are transitory, then we get the counterintuitive outcome that the blockholder buys shares (figure 5(b)) despite holding them is personally more costly. Because the increment in the blockholder stake is partly attributed by the market to a positive productivity shock (the term $1 - \beta_q Q_b / Q_\zeta$ in table 1), the blockholder can benefit from the increment in price when he reverts his position after the shock, so the subsequent trading gain offsets the higher cost of holding the shares at the time of the shock. This trading strategy is not significantly costly, when liquidity shocks are transitory.

Finally, Figures 4(c) and 5(c) show the response to dividend shocks. In the absence of liquidity shocks, cash flows are uninformative, the market has nothing to learn from cash flows, and there is

no reaction to cash flow shocks. This is no longer the case under liquidity shocks because trading is not fully revealing, so cash flows are informative. Because the market incorrectly attributes a transitory cash flow shock to variation in ability or liquidity needs, it expects the blockholder to buy more shares which in turn leads to an increase in the stock price.



(a) Impulse response function for shocks to ζ_t

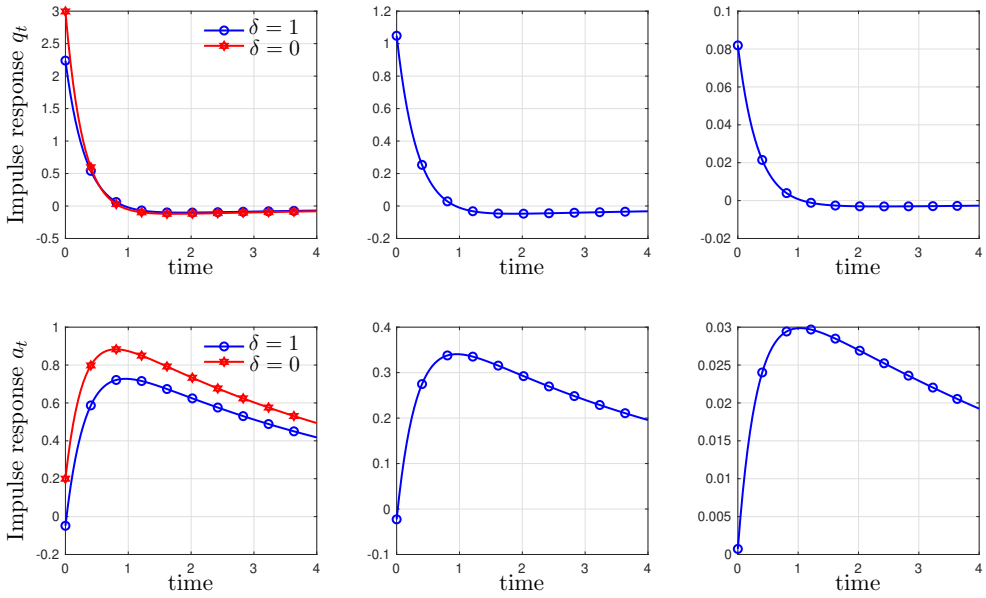
(b) Impulse response function for shocks to b_t

(c) Impulse response function for shocks to D_t

Figure 4: Impulse response trading and effort with transitory liquidity shocks ($\lambda = 0.1$). Parameters: $\gamma_M = 1$, $\gamma_L = 10$, $\sigma_D = 1$, $\sigma_\zeta = 0.2$, $\sigma_b = 0.2$, $\kappa = 0.2$, $\phi = 0.5$, $\psi = 1$, $r = 0.05$, $\mu_D = 1$.

7 Empirical Implications

We find that under information asymmetry the blockholder faces an illiquid market and holds a larger block, so information asymmetry leads to greater ownership concentration. However, we show that asymmetric information has a long-term impact on ownership only if there is a risk premium associated to the blockholder private information. If there is no such risk premium, for example because ability shocks can be diversified, then asymmetric information only has temporary effects. We show that this effect is more acute in volatile environments with large uncertainty about blockholder ability, and in settings where the risk-bearing capacity of the market is limited. This prediction is seemingly consistent with the conventional wisdom that more opaque markets (e.g., in under-developed countries) are characterized by greater ownership concentration, relative to U.S.



(a) Impulse response function for shocks to ζ_t

(b) Impulse response function for shocks to b_t

(c) Impulse response function for shocks to D_t

Figure 5: Impulse response trading and effort with persistent liquidity shocks ($\lambda = 0.5$). Parameters: $\gamma_M = 1$, $\gamma_L = 10$, $\sigma_D = 1$, $\sigma_\zeta = 0.2$, $\sigma_b = 0.2$, $\kappa = 0.2$, $\phi = 0.5$, $\psi = 1$, $r = 0.05$, $\mu_D = 1$.

However, Holderness (2007) questions the notion that firm ownership is relatively diffuse in the U.S. He finds that on average the large shareholders in a firm collectively own 39% (median 37%) of the voting power of the common stock. When a firm has at least one blockholder, 96% of the sample, the average size of the largest block is 26% (median 17%). He also finds an inverse relation between ownership concentration and firm size.

Because the blockholder stake affects the firm's productivity, asymmetric information has an impact on firm productivity. Thus, asymmetric information can lead to higher productivity. The impact of asymmetric information is more subtle in the presence of unobservable liquidity shocks. Even though asymmetric information still increases productivity in the long-run, it might also decrease productivity in the short run due to the Ratchet effect because the blockholder has incentives to manipulate short term prices. The evidence with respect to productivity improvements of blockholders is mixed. Barclay and Holderness (1991) find that trade of large blocks between investors lead to 16% increase in market value. Similarly, looking at a broader class of blockholders (investors holding more than 5% of the shares), Cronqvist and Fahlenbrach (2008) show significant blockholder fixed effects in operational, financing, and compensation policies of a firm. On the other hand, Holderness and Sheehan (1988) finds that diffuse ownerwhip makes no difference for Tobin's Q. In the contexts of managerial ownership, Fabisik, Fahlenbrach, Stulz, and Taillard (2018) finds a

negative relation between managerial ownership and performance. Consistent with our model, they find that the relation between ownership and productivity is driven by market liquidity. According to Fabisik et al., the negative relation between firm productivity and ownership is driven by the higher liquidity of more productive firms, which allows manager to divest. Consistent with their hypothesis, they find that firms with high managerial ownership corresponds to those with low liquidity, which tend to be the less productive ones, and this would help to explain the negative relation between managerial ownership and profitability measures such as the Tobin's Q.

In the case of activist hedge funds. Denes et al. (2017) finds that 8 of 11 studies on hedge fund activism conclude that earnings-based measures of operating performance improve after activist interventions, and the remaining three find no change. Brav et al. (2008) finds that activists target factories that experience abnormal declines in productivity in the years preceding the activist intervention, followed by productivity increases afterward. Finally, Brav et al. (2015) uses plant-level data from manufacturing firms to assess the operational effects of hedge fund activism. The biggest improvements in productivity are concentrated among plants that were sold after the activist intervention. deHaan et al. (2018) confirm prior findings that the operating performance of target firms appears to improve after an intervention when compared to control firms that are matched on the level but not trend in pre-activism ROA.

In our model, an unintended consequence of asymmetric information is a greater cash-flow volatility. In the presence of asymmetric information the blockholder sell shares in reaction to negative productivity shocks, which amplifies the impact on cash flows. This means that the firm is more exposed to variation in the blockholder's ability, which should lead to more risky cash flows. One empirical prediction then is that market illiquidity generated by asymmetric information should increase the volatility of cash flows. We are unaware of empirical evidence looking at this particular effect; however, the literature has documented an association between return volatility and ownership concentration that is consistent with our model. Two explanations have been advanced for why stock-return volatility might affect ownership concentration. Demsetz and Lehn (1985) propose that the greater the volatility of a firm's environment, the more difficult it is for outsiders to monitor management, and the greater are the benefits of inside ownership. In other words, according to Demsetz and Lehn volatility is the caused of concentration rather than its consequence. Alternatively, Himmelberg et al. (1999) look at the impact on volatility in light of risk aversion. Because large shareholders may be underdiversified as a result of their block investment, the optimal level of block ownership should decline, ceteris paribus, as volatility increases. Our model suggest that the relation between ownership concentration and cash flow volatility crucially depends on the liquidity of the market. We predict that the relation between cash flow volatility and ownership concentration should be concentrated in shares that suffer from

price impact (that is, illiquid shares). In our model, higher volatility increases concentration only if productivity shocks are private information and cannot be diversified; moreover, unlike in Demsetz and Lehn's explanation this happens not because monitoring is more valuable but because it is more costly for the blockholder to divest. In other words, our model predict that the relation between cash flow volatility and ownership should be stronger in firms that are more opaque, and in which asymmetric information is likely to be more important.

8 Conclusion

This paper studies the impact strategic trading on blockholder ownership and firm productibity. We propose a model where a blockholder can affects the value of the firm value but has private information about the effect of his interventions. We contrast the case where ability is observable with the one in which ability is private information. Asymmetric information generates price impact which allows us to study the impact of liquidity on trading and long-term ownership. We show that without information asymmetry, the blockholder's trading is characterized by the same Coasian dynamics previously identified in the literature. In this context, an improvement in the blockholder's productivity increases the price and induces the blockholder to reduce his holdings. Effectively, the incentive of the blockholder to hedge against productivity shocks leads the blockholder to sell when he is most productive. By doing so, the blockholder effectively deprives other shareholders from some of the potential benefits of his activism.

We show that the blockholder's behavior drastically changes under information asymmetry. Order flow becomes informative when the market does not observe blockholder ability, so the blockholder trades gradually to mitigate the impact on prices. Furthermore, unlike in the absence of asymmetric information, the blockholder responds to a positive productivity shock by acquiring more shares. In addition to the impact on short-term ownership, we identify condition under which asymmetric information has a long run impact on outcomes. Notably, we show that if these shocks cannot be diversified, then the presence of information asymmetry modifies the firm's ownership structure causing the blockholder to hold a larger stake.

We also consider the incentives of blockholders to distort cash flows when prices are not fully revealing because of liquidity motivated trade. In this case, blockholders over-provide effort if they expect to sell shares in near future, and they under-provide effort effort if they expect to acquire shares. This effect resembles the ratchet effect previously identified in the literature on career concerns.

The literature has focused on whether/how liquidity interacts with activism (Maug (1998)). In our setting, the presence of information asymmetry reduces market's liquidity in that the order

flow has a price impact. Such illiquidity effect has some positive effects, insofar as it induces the blockholder to hold a larger stake. As such, the information asymmetry restores the incentive of the blockholder to hold an undiversified portfolio. In the long-run, this leads to more activism, and a higher firm productivity, but it also exacerbates the cash flow volatility, causing a higher risk premium. Our model has a number of limitations. First, we model intervention (effort by a manager or intervention by a blockholder) as having only short-term effects but, in practice, this has persistent effects on the firm's cash flows. Relaxing this assumption would be useful if one wishes to understand how policy makers should address blockholder's myopia, namely the blockholder's tendency to underestimate the long-run consequences of their interventions. Second, we assume a blockholder holdings are observable. In practice, their holdings are observed with some delay. For example, the Williams Act of 1968 requires that investors must disclose ownership stakes of more than 5% within 10 days, while in Britain investors must disclose stakes of more than 5% within two days. Third, our model captures the interventions of a blockholder in a stationary environment where the average holdings of the blockholder are positive. This is reasonable if we consider founders of a company, CEOs, private individuals, and institutional investors. By contrast, such an assumption is not realistic if we consider activist hedge funds whose intervention take place over a limited period of time, and they are not meant to last forever. In future work, it would be interesting to consider a model in which the activist investor decides the optimal timing to start acquiring shares as well as the optimal timing to exit its investment.

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Appendix

A Observable Case

Lemma A.1. *The market makers certainty equivalent is given by*

$$H(Y, X, \hat{\zeta}) = h_y Y + h_{yx} XY + h_{y\zeta} Y\hat{\zeta} + h_{yy} Y^2$$

such that

$$h_y = \frac{\mu_D}{r} + \frac{A_x Q_0}{r(r+Q_x)} \qquad h_{yx} = \frac{A_x}{r+Q_x}$$
$$h_{y\zeta} = \frac{1}{r+\kappa}\left(A_\zeta + \frac{A_x Q_\zeta}{r+Q_x}\right) \qquad h_{yy} = -\frac{\gamma_M}{2}\left(\sigma_D^2 + \sigma_\zeta^2 h_{y\zeta}^2\right).$$

Proof. The HJB equation is

$$rJ = \max_{c,q} u_M(c) + \left(rW - c - p(X,\hat{\zeta})q + Y(\mu_D + A_x X + A_\zeta \hat{\zeta})\right) J_W$$
$$+ qJ_Y + (Q_0 - Q_x X + Q_\zeta \hat{\zeta}) J_X - \kappa\hat{\zeta} J_\zeta + \frac{1}{2}\left(Y^2\sigma_D^2 J_{WW} + \sigma_\zeta^2 J_{\zeta\zeta}\right)$$

The first condition for the consumption choice is

$$\frac{\partial u_M(c)}{\partial c} = J_W,$$

and using our conjectured value function J we get

$$u_M(c) = rJ$$
$$c = r(W + H(Y, \hat{\zeta}, X))$$

Substituting in the HJB equation

$$rH = \max_q (\mu_D + A_x X + A_\zeta \hat{\zeta})Y - p(X,\hat{\zeta})q - \frac{1}{2}\left(r\gamma_M Y^2\sigma_D^2 + r\gamma_M\sigma_\zeta^2 H_\zeta^2\right)$$
$$qH_y + (Q_0 - Q_x X + Q_\zeta\hat{\zeta})H_x - \kappa\hat{\zeta}H_\zeta + \frac{1}{2}\sigma_\zeta^2 H_{\zeta\zeta}$$

We conjecture a quadratic form for the certainty equivalent H

$$H(Y, \hat{\zeta}, X) = h_y Y + h_{yx} XY + h_{y\zeta} Y\hat{\zeta} + h_{yy} Y^2$$

Substituting, we get

$$r(h_y Y + h_{yx} XY + h_{y\zeta} Y\hat{\zeta} + h_{yy} Y^2) = Y(\mu_D + A_x X + A_\zeta \hat{\zeta}) \\ + (Q_0 - Q_x X + Q_\zeta \hat{\zeta}) h_{yx} Y - \kappa h_{y\zeta} Y\hat{\zeta} - \frac{r\gamma_M}{2}\left(\sigma_D^2 + \sigma_\zeta^2 h_{y\zeta}^2\right) Y^2$$

We get the following system of equations

$$\begin{aligned} r h_y &= \mu_D + Q_0 h_{yx} \\ r h_{yx} &= A_x - Q_x h_{yx} \\ r h_{y\zeta} &= A_\zeta + Q_\zeta h_{yx} - \kappa h_{y\zeta} \\ r h_{yy} &= -\frac{r\gamma_M}{2}\left(\sigma_D^2 + \sigma_\zeta^2 h_{y\zeta}^2\right) \end{aligned}$$

Solving the system we get

$$\begin{aligned} h_y &= \frac{\mu_D}{r} + \frac{A_x Q_0}{r(r+Q_x)} \\ h_{yx} &= \frac{A_x}{r+Q_x} \\ h_{y\zeta} &= \frac{1}{r+\kappa}\left(A_\zeta + \frac{A_x Q_\zeta}{r+Q_x}\right) \\ h_{yy} &= -\frac{\gamma_M}{2}\left(\sigma_D^2 + \sigma_\zeta^2 h_{y\zeta}^2\right). \end{aligned}$$

□

Lemma A.2. *The large shareholder's certainty equivalent in the observable case is given by*

$$G^o(X,\zeta) = g_0^o + g_x^o X + g_{xx}^o X^2 + g_{\zeta\zeta}^o \zeta^2 + g_{x\zeta}^o X\zeta,$$

where

$$g_0^o = \frac{\sigma_\zeta^2}{r} g_{\zeta\zeta}^o \qquad\qquad g_x^o = \frac{\mu_D}{r}$$
$$g_{x\zeta}^o = \frac{\psi}{2\phi(r+\kappa+2r\gamma_L\sigma_\zeta^2 g_{\zeta\zeta}^o)} \qquad\qquad g_{xx}^o = \frac{1}{4r\phi} - \frac{\gamma_L}{2}\left(\sigma_D^2 + \sigma_\zeta^2 (g_{x\zeta}^o)^2\right)$$
$$g_{\zeta\zeta}^o = \frac{\pm\sqrt{(r+2\kappa)^2 + 2r\gamma_L\sigma_\zeta^2\frac{\psi^2}{\phi}} - (r+2\kappa)}{4r\gamma_L\sigma_\zeta^2}$$

The maximal certainty equivalent corresponds to the positive root $g_{\zeta\zeta}^o$.

Proof. Substituting our conjecture for the certainty equivalent in (4), we get the following system of for the coefficients in G.

$$rg_0^o = \sigma_\zeta^2 g_{\zeta\zeta}^o \tag{A.1}$$
$$rg_x^o = \mu_D \tag{A.2}$$
$$(r+\kappa)g_{x\zeta}^o = \frac{\psi}{2\phi} - (2r\gamma_L\sigma_\zeta^2 g_{\zeta\zeta}^o)g_{x\zeta}^o \tag{A.3}$$
$$(r+2\kappa)g_{\zeta\zeta}^o = -2r\gamma_L\sigma_\zeta^2(g_{\zeta\zeta}^o)^2 + \frac{\psi^2}{4\phi} \tag{A.4}$$
$$rg_{xx}^o = \frac{1}{4\phi} - \frac{r\gamma_L}{2}\left(\sigma_D^2 + \sigma_\zeta^2(g_{x\zeta}^o)^2\right) \tag{A.5}$$

From here, we immediately get that $g_x = \mu_D/r$ and

$$g_{\zeta\zeta}^o = \frac{\pm\sqrt{(r+2\kappa)^2 + 2r\gamma_L\sigma_\zeta^2\frac{\psi^2}{\phi}} - (r+2\kappa)}{4r\gamma_L\sigma_\zeta^2}$$

The rest of the expression follow directly. To verify that $G_+^o(X,\zeta) > G_-^o(X,\zeta)$, notice that, because $g_{0+}^o > v = g_{0-}^o$, we have that $g_{0+}^o + g_{x+}^o X > g_{0-}^o + g_{x-}^o X$, where $g_{\cdot+}^o$ and $g_{\cdot-}^o$ are the coefficients of G_+^o and G_-^o respectively. Next, let

$$M \equiv \begin{pmatrix} g_{\zeta\zeta+}^o - g_{\zeta\zeta-}^o & \frac{1}{2}(g_{x\zeta+}^o - g_{x\zeta-}^o) \\ \frac{1}{2}(g_{x\zeta+}^o - g_{x\zeta-}^o) & g_{xx+}^o - g_{xx-}^o. \end{pmatrix}$$

be the difference in the quadratic coefficients G^o_+ and G^o_-. The eigenvalues of M are 0 and

$$\frac{\sqrt{\frac{2r\sigma_\zeta^2\psi^2\gamma_L}{\phi}+(r+2\kappa)^2}\left(r\sigma_\zeta^2\psi^2\gamma_L\left(r\sigma_\zeta^2\left(\psi^2+1\right)\gamma_L+4\kappa\phi(r+\kappa)\right)+4\kappa^2\phi^2(r+\kappa)^2\right)}{2r\gamma_L\left(r\sigma_\zeta^3\psi^2\gamma_L+2\kappa\sigma_\zeta\phi(r+\kappa)\right)^2}>0,$$

which means that M is positive semidefinite. It follows that $(\zeta, X)M(\zeta, X)^\intercal \geq 0$, which means that $g^o_{xx+}X^2 + g^o_{\zeta\zeta+}\zeta^2 + g^o_{x\zeta+}X\zeta \geq g^o_{xx-}X^2 + g^o_{\zeta\zeta-}\zeta^2 + g^o_{x\zeta-}X\zeta$ for all (X, ζ).

□

Proof Proposition 1

Proof. Using the certainty equivalent for the blockholder, together with the first order condition we get that coefficients in the price function are

$$\begin{aligned}
P_0 &= g^o_x \\
P_x &= 2g^o_{xx} \\
P_\zeta &= g^o_{x\zeta}.
\end{aligned}$$

Moreover, from the solution of the market makers problem we also have that the coefficients are given by

$$\begin{aligned}
P_0 &= h_y + 2h_{yy} \\
P_x &= h_{yx} - 2h_{yy} \\
P_\zeta &= h_{y\zeta},
\end{aligned}$$

where

$$\begin{aligned}
h_y &= \frac{\mu_D}{r} + \frac{Q^o_0}{2\phi(r+Q^o_x)} \\
h_{yx} &= \frac{1}{2r\phi(r+Q^o_x)} \\
h_{y\zeta} &= \frac{1}{r+\kappa}\left(\frac{\psi}{2\phi} + \frac{Q^o_\zeta}{2\phi(r+Q^o_x)}\right) \\
h_{yy} &= -\frac{\gamma_M}{2}\left(\sigma_D^2 + \sigma_\zeta^2 h^2_{y\zeta}\right).
\end{aligned}$$

That is, in equilibrium, the marginal valuation of the large shareholder and the one of the competitive investors must coincide. Matching coefficients, we get

$$g_x^o = \frac{\mu_D}{r} + \frac{Q_0^o}{2\phi r(r+Q_x^o)} - 2\frac{\gamma_M}{2}\left(\sigma_D^2 + \sigma_\zeta^2 h_{y\zeta}^2\right) \tag{A.6}$$

$$2g_{xx}^o = \frac{A_x}{r+Q_x^o} + \gamma_M\left(\sigma_D^2 + \sigma_\zeta^2 h_{y\zeta}^2\right) \tag{A.7}$$

$$g_{x\zeta}^o = \frac{1}{r+\kappa}\left(A_\zeta + \frac{A_x Q_\zeta^o}{r+Q_x^o}\right) \tag{A.8}$$

We can solve for Q_x^o, Q_ζ^o using equations (A.7) and (A.8)

$$\frac{1}{r\phi} - \gamma_L\left(\sigma_D^2 + \sigma_\zeta^2 (g_{x\zeta}^{\text{obs}})^2\right) = \frac{2}{2\phi(r+Q_x^o)} + \gamma_M\left(\sigma_D^2 + \sigma_\zeta^2 h_{y\zeta}^2\right)$$

$$\frac{\psi}{2\phi} + \frac{Q_\zeta^o}{2\phi(r+Q_x^o)} = (r+\kappa)g_{x\zeta}^o,$$

which yields

$$Q_x^o = \frac{r^2\left(\gamma_L+\gamma_M\right)\left(\sigma_D^2 + \sigma_\zeta^2 (g_{x\zeta}^o)^2\right)}{(2\phi)^{-1} - r\left(\gamma_L+\gamma_M\right)\left(\sigma_D^2 + \sigma_\zeta^2 (g_{x\zeta}^o)^2\right)}$$

$$Q_\zeta^o = (r+Q_x^o)\left(2(r+\kappa)\phi(g_{x\zeta}^o)^2 - \psi\right)$$

For Q_0^o, we use the equation

$$g_x^o = \frac{\mu_D}{r} + \frac{Q_0^o}{2\phi r(r+Q_x^o)} - 2\frac{\gamma_M}{2}\left(\sigma_D^2 + \sigma_\zeta^2 h_{y\zeta}^2\right),$$

and substituting $g_\zeta^o = 0$ and $g_x^o = \frac{\mu_D}{r}$, we get

$$0 = \frac{Q_0^o}{2\phi r(r+Q_x^o)} - 2\frac{\gamma_M}{2}\left(\sigma_D^2 + \sigma_\zeta^2 h_{y\zeta}^2\right),$$

so

$$Q_0^o = 2r\gamma_M\phi(r+Q_x^o)\left(\sigma_D^2 + \sigma_\zeta^2 h_{y\zeta}^2\right).$$

Substituting Q_x^o, we arrive to

$$Q_0^o = \frac{r^2\gamma_M\left(\sigma_D^2 + \sigma_\zeta^2 (g_{x\zeta}^o)^2\right)}{(2\phi)^{-1} - r\left(\gamma_L+\gamma_M\right)\left(\sigma_D^2 + \sigma_\zeta^2 (g_{x\zeta}^o)^2\right)}.$$

□

Proof Lemma 2

Proof. The derivation of the certainty equivalent for the large shareholder is similar to the one for market makers. If we conjecture the following quadratic function for the certainty equivalent

$$G(\zeta, X) = g_0 + g_x X + g_\zeta \zeta + g_{xx} X^2 + g_{\zeta\zeta}\zeta^2 + g_{x\zeta} X\zeta,$$

then we get that

$$\begin{aligned} a &= \frac{\psi\zeta + X}{2\phi} \\ q &= \frac{g_x - R_0 + (2g_{xx} - R_x)X + g_{x\zeta}\zeta}{2R_q}. \end{aligned}$$

Substituting in the HJB equation, and matching coefficients, we arrive to the system of equations in the Lemma. □

A.1 Proofs

Proof Proposition 2

Proof. From the first order condition, we get that the coefficients $(\mathbf{A}, \mathbf{Q})$ are

$$\begin{aligned} A_x &= \frac{1}{2\phi} \\ A_\zeta &= \frac{\psi}{2\phi} \\ Q_0 &= \frac{g_x - R_0}{2R_q} \\ Q_x &= \frac{R_x - 2g_{xx}}{2R_q} \\ Q_\zeta &= \frac{g_{x\zeta}}{2R_q} \end{aligned}$$

Substituting our guess for the certainty equivalent in the HJB equation and matching coefficients, we get the following system of equation

$$rg_0 = \frac{g_x(g_x - 2R_0)}{4R_q} + \frac{R_0^2}{4R_q} + \sigma_\zeta^2 g_{\zeta\zeta} - \frac{1}{2} r\sigma_\zeta^2 \gamma_L g_\zeta^2 \tag{A.9a}$$

$$rg_x = \mu_D - r\gamma_L \sigma_\zeta^2 g_\zeta g_{x\zeta} + \frac{(R_0 - g_x)(R_x - 2g_{xx})}{2R_q} \tag{A.9b}$$

$$(r+\kappa) g_\zeta = \frac{g_{x\zeta}(g_x - R_0)}{2R_q} - 2r\gamma_L \sigma_\zeta^2 g_\zeta g_{\zeta\zeta} \tag{A.9c}$$

$$(r+\kappa) g_{x\zeta} = \frac{\psi}{2\phi} + \frac{g_{x\zeta}(2g_{xx} - R_x)}{2R_q} - 2r\gamma_L \sigma_\zeta^2 g_{\zeta\zeta} g_{x\zeta} \tag{A.9d}$$

$$(r+2\kappa) g_{\zeta\zeta} = \frac{\psi^2}{4\phi} + \frac{g_{x\zeta}^2}{4R_q} - 2r\gamma_L \sigma_\zeta^2 g_{\zeta\zeta}^2 \tag{A.9e}$$

$$rg_{xx} = \frac{1}{4\phi} + \frac{(2g_{xx} - R_x)^2}{4R_q} - \frac{1}{2} r\gamma_L \left(\sigma_D^2 + \sigma_\zeta^2 g_{x\zeta}^2\right) \tag{A.9f}$$

It is convenient to express the coefficients $\mathbf{R}$ in terms of the coefficients $\mathbf{Q}$

$$\begin{aligned} R_0 &= \frac{Q_\zeta g_x - Q_0 g_{x\zeta}}{Q_\zeta} \\ R_q &= \frac{g_{x\zeta}}{2Q_\zeta} \\ R_x &= 2g_{xx} + \frac{Q_x}{Q_\zeta} g_{x\zeta} \end{aligned}$$

Substituting this in (A.9a)- (A.9f) , we get the system

$$rg_0 = -\frac{1}{2} r\gamma_L g_\zeta^2 \sigma_\zeta^2 + \frac{1}{2}\frac{Q_0^2}{Q_\zeta} g_{x\zeta} + \sigma_\zeta^2 g_{\zeta\zeta} \tag{A.10a}$$

$$rg_x = \mu_D - \frac{Q_0 Q_x}{Q_\zeta} g_{x\zeta} - r\gamma_L \sigma_\zeta^2 g_\zeta g_{x\zeta} \tag{A.10b}$$

$$(r+\kappa) g_\zeta = -2r\gamma_L \sigma_\zeta^2 g_{\zeta\zeta} g_\zeta + Q_0 g_{x\zeta} \tag{A.10c}$$

$$(r+\kappa) g_{x\zeta} = \frac{\psi}{2\phi} - (Q_x + 2r\gamma_L \sigma_\zeta^2 g_{\zeta\zeta}) g_{x\zeta} \tag{A.10d}$$

$$(r+2\kappa) g_{\zeta\zeta} = \frac{Q_\zeta}{2} g_{x\zeta} - 2r\gamma_L \sigma_\zeta^2 g_{\zeta\zeta}^2 + \frac{\psi^2}{4\phi} \tag{A.10e}$$

$$rg_{xx} = \frac{Q_x^2}{2Q_\zeta} g_{x\zeta} + \frac{1}{4\phi} - \frac{r\gamma_L}{2} \left(\sigma_D^2 + \sigma_\zeta^2 g_{x\zeta}^2\right) \tag{A.10f}$$

The next step is to find expressions for the coefficients $\mathbf{Q}$. The coefficients in the price function are

$$\begin{aligned} P_0 &= g_x - \frac{Q_0}{2Q_\zeta} g_{x\zeta} \\ P_x &= 2g_{xx} + \frac{Q_x}{Q_\zeta} g_{x\zeta} \\ P_\zeta &= \frac{g_{x\zeta}}{2}. \end{aligned}$$

But, from the solution of the market makers' problem we have that

$$\begin{aligned} P_0 &= h_y + 2h_{yy} \\ P_x &= h_{yx} - 2h_{yy} \\ P_\zeta &= h_{y\zeta}, \end{aligned}$$

where

$$\begin{aligned} h_y &= \frac{\mu_D}{r} + \frac{Q_0}{2\phi r(r+Q_x)} \\ h_{yx} &= \frac{1}{2\phi(r+Q_x)} \\ h_{y\zeta} &= \frac{1}{r+\kappa}\left(\frac{\psi}{2\phi} + \frac{Q_\zeta}{2\phi(r+Q_x)}\right) \\ h_{yy} &= -\frac{\gamma_M}{2}\left(\sigma_D^2 + \sigma_\zeta^2 h_{y\zeta}^2\right). \end{aligned}$$

Matching coefficients

$$\begin{aligned} 2g_{xx} + \frac{Q_x}{Q_\zeta} g_{x\zeta} &= \frac{1}{2\phi(r+Q_x)} + \gamma_M\left(\sigma_D^2 + \sigma_\zeta^2 \frac{g_{x\zeta}^2}{4}\right) \\ g_{x\zeta} &= \frac{1}{r+\kappa}\left(\frac{\psi}{\phi} + \frac{Q_\zeta}{\phi(r+Q_x)}\right) \end{aligned}$$

The following block of equations can be solved independently

$$(r+\kappa)g_{x\zeta} = \frac{\psi}{2\phi} - (Q_x + 2r\gamma_L\sigma_\zeta^2 g_{\zeta\zeta})g_{x\zeta} \tag{A.11}$$

$$(r+2\kappa)g_{\zeta\zeta} = \frac{Q_\zeta}{2}g_{x\zeta} - 2r\gamma_L\sigma_\zeta^2 g_{\zeta\zeta}^2 + \frac{\psi^2}{4\phi} \tag{A.12}$$

$$rg_{xx} = \frac{Q_x^2}{2Q_\zeta}g_{x\zeta} + \frac{1}{4\phi} - \frac{r\gamma_L}{2}\left(\sigma_D^2 + \sigma_\zeta^2 g_{x\zeta}^2\right) \tag{A.13}$$

$$2g_{xx} + \frac{Q_x}{Q_\zeta}g_{x\zeta} = \frac{1}{2\phi(r+Q_x)} + \gamma_M\left(\sigma_D^2 + \sigma_\zeta^2\frac{g_{x\zeta}^2}{4}\right) \tag{A.14}$$

$$g_{x\zeta} = \frac{1}{r+\kappa}\left(\frac{\psi}{\phi} + \frac{Q_\zeta}{\phi(r+Q_x)}\right) \tag{A.15}$$

Finally, using equations (A.13) and (A.14) to eliminate g_{xx} we arrive to

$$(r+\kappa)g_{x\zeta} = \frac{\psi}{2\phi} - (Q_x + 2r\gamma_L\sigma_\zeta^2 g_{\zeta\zeta})g_{x\zeta} \tag{A.16}$$

$$(r+2\kappa)g_{\zeta\zeta} = \frac{Q_\zeta}{2}g_{x\zeta} - 2r\gamma_L\sigma_\zeta^2 g_{\zeta\zeta}^2 + \frac{\psi^2}{4\phi} \tag{A.17}$$

$$\frac{Q_x}{2Q_\zeta}\left(r+2Q_x\right)g_{x\zeta} = -\frac{Q_x}{2\phi(r+Q_x)} + r(\gamma_L+\gamma_M)\sigma_D^2 + r\left(\frac{\gamma_M}{4}+\gamma_L\right)\sigma_\zeta^2 g_{x\zeta}^2 \tag{A.18}$$

$$g_{x\zeta} = \frac{1}{r+\kappa}\left(\frac{\psi}{\phi} + \frac{Q_\zeta}{\phi(r+Q_x)}\right). \tag{A.19}$$

Replacing (A.19) in (A.16) we get a an equation that is linear in $g_{\zeta\zeta}$. Solving for $g_{\zeta\zeta}$ and replacing in (A.17) we end with a system of three equations in $Q_x, Q_\zeta, g_{x\zeta}$. Substituting $g_{x\zeta} = 2R_qQ_\zeta$ and simplifying terms we get

$$2R_q^2\mathcal{Q}_\zeta(R_q,Q_x)^2\left[r\gamma_L\sigma_\zeta^2\left(4R_q\mathcal{Q}_\zeta(R_q,Q_x)^2 + \frac{\psi^2}{\phi}\right) + 2\kappa(r+\kappa) - 2Q_x\left(r+Q_x\right)\right] \tag{A.20}$$
$$+ R_q\mathcal{Q}_\zeta(R_q,Q_x)(r+2Q_x)\frac{\psi}{\phi} - \frac{\psi^2}{4\phi^2} = 0$$

$$2r\phi(r+Q_x)\left[\left(\gamma_L+\gamma_M\right)\sigma_D^2 + R_q^2\mathcal{Q}_\zeta(R_q,Q_x)^2\left(4\gamma_L+\gamma_M\right)\sigma_\zeta^2\right] \tag{A.21}$$
$$- Q_x\left[rR_q + 1 + 6r\phi R_qQ_x + 4\phi R_qQ_x^2\right] = 0$$

where

$$\mathcal{Q}_\zeta(R_q,Q_x) = \frac{\psi}{2(r+\kappa)\phi R_q - (r+Q_x)^{-1}}$$

Finally, we compute the steady state holdings $\bar{X}_{ss}$. Using the envelope theorem in equation (11)

we get

$$rG_x = (\mu_D + A_x X + A_\zeta \zeta) - R_x q^L(X,\zeta) - r\gamma_L\left(\sigma_D^2 X + \sigma_\zeta^2 G_\zeta G_{x\zeta}\right) + qG_{xx} - \kappa\zeta G_{x\zeta} + \frac{1}{2}\sigma_\zeta^2 G_{x\zeta\zeta},$$

which has the stochastic representation

$$G_x = E_t\left[\int_t^\infty e^{-r(s-t)}\left(\mu_D + A_x X_s + A_\zeta \zeta_s - q_s^L R_x - r\gamma_L\sigma_\zeta^2 g_{x\zeta}\left(g_\zeta + 2g_{\zeta\zeta}\zeta_s\right) - r\gamma_L\left(\sigma_D^2 + \sigma_\zeta^2 g_{x\zeta}^2\right) X_s\right) ds\right]$$

Similarly,

$$H_y = E_t\left[\int_t^\infty e^{-r(s-t)}\left(\mu_D - r\gamma_M\left(\sigma_D^2 + \sigma_\zeta^2 h_{y\zeta}^2\right)(1 - X_s)\right) ds\right]$$

In steady state $\mathbb{E}[G_x] = \mathbb{E}[H_y]$ and $\mathbb{E}[\zeta_t] = \mathbb{E}[q_t^L] = 0$ so

$$\gamma_L\sigma_\zeta^2 g_{x\zeta} g_\zeta + \gamma_L\left(\sigma_D^2 + \sigma_\zeta^2 g_{x\zeta}^2\right)\bar{X}_{ss} = \gamma_M\left(\sigma_D^2 + \sigma_\zeta^2 h_{y\zeta}^2\right)(1 - \bar{X}_{ss}),$$

which it is equal to

$$\gamma_L\sigma_\zeta^2 g_{x\zeta} g_\zeta + \gamma_L\left(\sigma_D^2 + \sigma_\zeta^2 g_{x\zeta}^2\right)\bar{X}_{ss} = \gamma_M\left(\sigma_D^2 + \sigma_\zeta^2\frac{g_{x\zeta}^2}{4}\right)(1 - \bar{X}_{ss}) \tag{A.22}$$

Multiplying equation (A.10c) by $g_{x\zeta}$ we get

$$\gamma_L\sigma_\zeta^2 g_\zeta g_{x\zeta} = \frac{\gamma_L\sigma_\zeta^2 Q_0}{r + \kappa + 2r\gamma_L\sigma_\zeta^2 g_{\zeta\zeta}} g_{x\zeta}^2 = \frac{\gamma_L\sigma_\zeta^2 Q_x}{r + \kappa + 2r\gamma_L\sigma_\zeta^2 g_{\zeta\zeta}} g_{x\zeta}^2 \bar{X}_{ss} \tag{A.23}$$

Substituting in equation (A.22) and solving for $\bar{X}_{ss}$ we get

$$\bar{X}_{ss} = \frac{\gamma_M}{\omega(R_q, Q_x)\gamma_L + \gamma_M} \tag{A.24}$$

where

$$\omega(R_q, Q_x) \equiv \frac{\sigma_D^2 + 4\psi\sigma_\zeta^2 R_q^2 \mathcal{Q}_\zeta(R_q, Q_x)^2\Big(\psi - 4\phi\mathcal{Q}_\zeta(R_q, Q_x)Q_x R_q\Big)^{-1}}{\sigma_D^2 + \sigma_\zeta^2 R_q^2 \mathcal{Q}_\zeta(R_q, Q_x)^2} \tag{A.25}$$

□

Proof Proposition 4

Proof. We consider the limit when $\sigma_\zeta^\epsilon = \sqrt{\epsilon}\sigma_\zeta$, $\kappa^\epsilon = \epsilon\kappa$, and ϵ goes to zero. Taking the limit in the polynomial system in Proposition 2 we get

$$-4R_q^2\mathcal{Q}_\zeta(R_q,Q_x)^2Q_x\left(r+Q_x\right)+R_q\mathcal{Q}_\zeta(R_q,Q_x)(r+2Q_x)\frac{\psi}{\phi}-\frac{\psi^2}{4\phi^2}=0 \tag{A.26}$$

$$2r\phi(r+Q_x)\left(\gamma_L+\gamma_M\right)\sigma_D^2-Q_x\left[rR_q+1+6r\phi R_qQ_x+4\phi R_qQ_x^2\right]=0 \tag{A.27}$$

The previous system has only one positive solution

$$Q_x=\frac{\sqrt{r\phi\left(\gamma_L+\gamma_M\right)\sigma_D^2\left(9r\phi\left(\gamma_L+\gamma_M\right)\sigma_D^2+2\right)+1}-\left(1+r\phi\left(\gamma_L+\gamma_M\right)\sigma_D^2\right)}{4\phi\left(\gamma_L+\gamma_M\right)\sigma_D^2}$$

$$R_q=\frac{\sqrt{r\phi\sigma_D^2\left(\gamma_L+\gamma_M\right)\left(9r\phi\left(\gamma_L+\gamma_M\right)\sigma_D^2+2\right)+1}+\left(1+r\phi\left(\gamma_L+\gamma_M\right)\sigma_D^2\right)}{4r^2\phi}.$$

Substituting in $\mathcal{Q}_\zeta(R_q,Q_x)$ we get

$$Q_\zeta=\frac{\psi}{2\phi\left(\gamma_L+\gamma_M\right)\sigma_D^2}$$

Next, to derive the limit for $\bar{X}_{ss}$, we consider the limit of $\omega(R_q,Q_x)$, which is given by

$$\omega(R_q,Q_x)\equiv\frac{\sigma_D^2+4\psi\sigma_\zeta^2R_q^2\mathcal{Q}_\zeta(R_q,Q_x)^2\Big(\psi-4\phi\mathcal{Q}_\zeta(R_q,Q_x)Q_xR_q\Big)^{-1}}{\sigma_D^2+\sigma_\zeta^2R_q^2\mathcal{Q}_\zeta(R_q,Q_x)^2}$$

If we substitute the limit for (Q_x,Q_ζ,R_q) that we found above we get that

$$\lim\left(\psi-4\phi\mathcal{Q}_\zeta(R_q,Q_x)Q_xR_q\right)=0,$$

which means that we have to consider the limit of the ratio

$$\frac{2\sigma_\zeta^2Q_\zeta R_q}{\psi-4\phi\mathcal{Q}_\zeta(R_q,Q_x)Q_xR_q}.$$

Using the fact that $2Q_\zeta R_q=g_{x\zeta}$, it is convenient to rewrite the previous expression as

$$\frac{1}{2\phi}\frac{\sigma_\zeta^2g_{x\zeta}}{\frac{\psi}{2\phi}-g_{x\zeta}Q_x}$$

Equation (A.11) implies that

$$(r+\kappa+2r\gamma_L\sigma_\zeta^2 g_{\zeta\zeta})g_{x\zeta}=\frac{\psi}{2\phi}-g_{x\zeta}Q_x,$$

which means that

$$\frac{\sigma_\zeta^2 g_{x\zeta}}{\frac{\psi}{2\phi}-g_{x\zeta}Q_x}=\frac{\sigma_\zeta^2}{r+2\kappa+2r\gamma_L\sigma_\zeta^2 g_{\zeta\zeta}-\kappa}.$$

From equation (A.12) we get that

$$r+2\kappa+2r\gamma_L\sigma_\zeta^2 g_{\zeta\zeta}=\frac{\frac{Q_\zeta}{2}g_{x\zeta}+\frac{\psi^2}{4\phi}}{g_{\zeta\zeta}}$$

so

$$\frac{\sigma_\zeta^2 g_{x\zeta}}{\frac{\psi}{2\phi}-g_{x\zeta}Q_x}=\frac{2\sigma_\zeta^2 g_{\zeta\zeta}}{Q_\zeta g_{x\zeta}+\frac{\psi^2}{2\phi}-2\kappa g_{\zeta\zeta}}.$$

Letting $\hat{g}_{\zeta\zeta}\equiv\sigma_\zeta^2 g_{\zeta\zeta}$ and remembering that $\bar{\sigma}_\zeta^2=\sigma_\zeta^2/2\kappa$ we get

$$\frac{\sigma_\zeta^2 g_{x\zeta}}{\frac{\psi}{2\phi}-g_{x\zeta}Q_x}=\frac{2\hat{g}_{\zeta\zeta}}{Q_\zeta g_{x\zeta}+\frac{\psi^2}{2\phi}-(\bar{\sigma}_\zeta^2)^{-1}\hat{g}_{\zeta\zeta}}.$$

The only step left is to determine the limit of $\hat{g}_{\zeta\zeta}$. From equation (A.12) we get

$$(r+2\kappa+2r\gamma_L\hat{g}_{\zeta\zeta})\hat{g}_{\zeta\zeta}=\sigma_\zeta^2\left(\frac{Q_\zeta}{2}g_{x\zeta}+\frac{\psi^2}{4\phi}\right).$$

Taking the limit when σ_ζ^2 and κ go to zero we get two solutions for $\hat{g}_{\zeta\zeta}$: $\hat{g}_{\zeta\zeta}=0$ and $\hat{g}_{\zeta\zeta}=-(2\gamma_L)^{-1}$. If we considering the non-zero solution we get

$$\lim\frac{\sigma_\zeta^2 g_{x\zeta}}{\frac{\psi}{2\phi}-g_{x\zeta}Q_x}=-\frac{\gamma_L^{-1}}{\lim Q_\zeta g_{x\zeta}+\frac{\psi^2}{2\phi}+(2\gamma_L\bar{\sigma}_\zeta^2)^{-1}},$$

where

$$\lim Q_\zeta g_{x\zeta}=\lim 2Q_\zeta^2 R_q=\frac{\psi^2}{\phi\left(\sqrt{(\alpha+1)^2+8\alpha^2}-(\alpha+1)\right)}=\frac{\psi^2}{4\phi^2\left(\gamma_L+\gamma_M\right)\sigma_D^2 Q_x}$$

From here we get that $\omega_0 \equiv \lim \omega(R_q, Q_x)$ is given by

$$\begin{aligned}
\omega_0 &= 1 + \frac{\psi}{\phi} R_q Q_\zeta \frac{\sigma_\zeta^2 g_{x\zeta}}{\frac{\psi}{2\phi} - g_{x\zeta} Q_x} \frac{1}{\sigma_D^2} \\
&= 1 - \frac{\psi}{\phi} \frac{2R_q Q_\zeta}{2\gamma_L \bar{\sigma}_\zeta^2 \left(\frac{\psi^2}{4\phi^2(\gamma_L+\gamma_M)\sigma_D^2 Q_x} + \frac{\psi^2}{2\phi} \right) + 1} \frac{\bar{\sigma}_\zeta^2}{\sigma_D^2} \\
&= 1 - \frac{\psi^2}{4r^2\phi^3 \left(\gamma_L + \gamma_M\right) \sigma_D^2} \frac{(\sqrt{(\alpha+1)^2+8\alpha^2}+\alpha+1)(\sqrt{(\alpha+1)^2+8\alpha^2}-\alpha-1)}{\frac{\psi^2}{\phi}\gamma_L \bar{\sigma}_\zeta^2 \left(\sqrt{(\alpha+1)^2+8\alpha^2} - \alpha + 1 \right) + \sqrt{(\alpha+1)^2+8\alpha^2} - \alpha - 1} \frac{\bar{\sigma}_\zeta^2}{\sigma_D^2} \\
&= 1 - \frac{2\left(\gamma_L + \gamma_M\right) \bar{\sigma}_\zeta^2}{\gamma_L \bar{\sigma}_\zeta^2 \left(\sqrt{(\alpha+1)^2+8\alpha^2} - \alpha + 1 \right) + \left(\frac{\psi^2}{\phi} \right)^{-1} \left(\sqrt{(\alpha+1)^2+8\alpha^2} - \alpha - 1 \right)},
\end{aligned}$$

which corresponds to the expression in the appendix. Because ω_0 is monotone in $\bar{\sigma}_\zeta^2$ it is enough to evaluate it at $\bar{\sigma}_\zeta^2 = 0$ and take the limit as $\bar{\sigma}_\zeta^2 \to \infty$ to verify that $\omega_0 \in (-\gamma_M/\gamma_L, 1]$.

□

Proof Proposition 3

Proof. Taking the limit when $\sigma_\zeta^2 \to 0$ in Proposition 2 we get

$$0 = \Big(2R_q\phi\left(r+Q_x\right)\left(r+3\kappa-2Q_x\right)-1\Big)\Big(2R_q\phi\left(r+Q_x\right)\left(r+\kappa+2Q_x\right)+1\Big) \tag{A.28}$$

$$0 = \left(2R_q\phi\left(r+Q_x\right)\left(r+\kappa\right)-1\right)\Big(Q_x\big(2R_q\phi\left(r+Q_x\right)\left(r+2Q_x\right)+1\big) - 2r\phi\sigma_D^2\left(r+Q_x\right)\left(\gamma_L+\gamma_M\right)\Big) \tag{A.29}$$

Equation (A.28) can be satisfied by positive (R_q, Q_x) only if

$$2R_q\phi\left(r+Q_x\right)\left(r+3\kappa-2Q_x\right)-1=0.$$

On the other hand,

$$Q_\zeta = \frac{\psi}{2(r+\kappa)\phi R_q - (r+Q_x)^{-1}}$$

is well defined only if the denominator is different than zero, which means that in equation (A.29) we can limit attention to

$$Q_x\big(2R_q\phi\left(r+Q_x\right)\left(r+2Q_x\right)+1\big) - 2r\phi\sigma_D^2\left(r+Q_x\right)\left(\gamma_L+\gamma_M\right) = 0$$

Hence, the coefficients (Q_x, R_q) can be found solving the following system of two equations.

$$\begin{aligned} 2R_q\phi\,(r+Q_x)\,(r+3\kappa-2Q_x)-1&=0\\ Q_x\big(2R_q\phi\,(r+Q_x)\,(r+2Q_x)+1\big)-2r\phi\sigma_D^2\,(r+Q_x)\,(\gamma_L+\gamma_M)&=0. \end{aligned}$$

Letting

$$\eta \equiv \frac{2r+3\kappa+2r\phi(r-3\kappa)\,(\gamma_L+\gamma_M)\,\sigma_D^2}{4}$$

we we can write the two solutions to the previous system as

$$\begin{aligned} Q_x^{(1)} &= \frac{-\eta+\sqrt{\eta^2+2r^3\phi^2(r+3\kappa)\,(\gamma_L+\gamma_M)^2\,\sigma_D^4}}{2r\phi\,(\gamma_L+\gamma_M)\,\sigma_D^2}\\ R_q^{(1)} &= \frac{\eta+\sqrt{\eta^2+2r^3\phi^2(r+3\kappa)\,(\gamma_L+\gamma_M)^2\,\sigma_D^4}}{r\phi(r+3\kappa)(2r+3\kappa)} \end{aligned}$$

and

$$\begin{aligned} Q_x^{(2)} &= -\frac{\eta+\sqrt{\eta^2+2r^3\phi^2(r+3\kappa)\,(\gamma_L+\gamma_M)^2\,\sigma_D^4}}{2r\phi\,(\gamma_L+\gamma_M)\,\sigma_D^2}\\ R_q^{(2)} &= \frac{\eta-\sqrt{\eta^2+2r^3\phi^2(r+3\kappa)\,(\gamma_L+\gamma_M)^2\,\sigma_D^4}}{r\phi(r+3\kappa)(2r+3\kappa)} \end{aligned}$$

Only the first solution is positive, so the equilibrium coefficients are

$$\begin{aligned} Q_x &= \frac{\sqrt{\eta^2+2r^3\phi^2(r+3\kappa)\,(\gamma_L+\gamma_M)^2\,\sigma_D^4}-\eta}{2r\phi\,(\gamma_L+\gamma_M)\,\sigma_D^2}\\ R_q &= \frac{\sqrt{\eta^2+2r^3\phi^2(r+3\kappa)\,(\gamma_L+\gamma_M)^2\,\sigma_D^4}+\eta}{r\phi(r+3\kappa)(2r+3\kappa)} \end{aligned}$$

The coefficient Q_ζ is given by

$$\frac{Q_\zeta}{Q_x} = \frac{1}{2}\frac{\psi(2r+3\kappa)}{\sqrt{\eta^2+2r^3\phi^2(r+3\kappa)\,(\gamma_L+\gamma_M)^2\,\sigma_D^4}-\eta-2r\phi\kappa\,(\gamma_L+\gamma_M)\,\sigma_D^2}$$

Hence, Q_ζ is positive only if

$$\phi > \frac{\kappa(2r+3\kappa)}{2r(r+\kappa)^2\,(\gamma_L+\gamma_M)\,\sigma_D^2}$$

Finally, we compute the steady state. In this case, we have that

$$\lim\left(\psi - 4\phi\mathcal{Q}_\zeta(R_q, Q_x)Q_x R_q\right) \neq 0,$$

so we can directly take the limit of $\omega(R_q, Q_x)$ as $\sigma_\zeta^2 \to 0$ to get that $\lim \omega(R_q, Q_x) = 1$, so

$$\bar{X}_{ss} = \frac{\gamma_M}{\gamma_L + \gamma_M}.$$

□

Proof Corollary 1

Proof. Consider the limit of the coefficients in Lemma 1 when $\sigma_\zeta^2 \to 0$ (regardless of whether $\kappa > 0$ or $\kappa \to 0$), which are given by

$$\begin{aligned} P_x &= \frac{A_x}{r + Q_x} \\ P_\zeta &= \frac{1}{r+\kappa}\left(A_\zeta + A_x \frac{Q_\zeta}{r + Q_x}\right). \end{aligned}$$

It can be verified that in the case 4 we have that $Q_x^o > Q_x^u$. On the other hand, in the case of the limit in Propositions 3, we can verify that $Q_x^o > Q_x^u$ if and only if κ is lower than some upper threshold $\bar{\kappa}$, which is provided in the proof of Proposition 7. These inequalities are verified as part of the proofs of Propositions 6 and 7. It follows directly that $P_x^u > P_x^o$. Similarly, we have that in the limit $Q_\zeta^o = 0$ and $Q_\zeta > 0$ as long as $\phi \geq \bar{\phi}$, where $\bar{\phi}$ is defined in Proposition 3. Thus, it follows that $P_\zeta^u > P_\zeta^o$ if and only if $\phi \geq \bar{\phi}$. □

Proof Proposition 5

Proof. It follows directly from Proposition 4 that ω_0 is decreasing in ψ and $\bar{\sigma}_\zeta^2$, which means that $\bar{X}_{ss}$ is increasing in these parameters.

From Lemma 1 we get that the mean steady state price is

$$\begin{aligned} \bar{p}_{ss} &= \frac{\mu_D}{r} - \gamma_M(1 - \bar{X}_{ss})\sigma_D^2 + \frac{\bar{X}_{ss}}{2\phi r} \\ &= \frac{\mu_D}{r} - \frac{\omega_0\gamma_L}{\omega_0\gamma_L + \gamma_M}\gamma_M\sigma_D^2 + \frac{1}{2\phi r}\frac{\gamma_M}{\omega_0\gamma_L + \gamma_M} \end{aligned}$$

The expected price is decreasing in ω_0; it follows that the price is decreasing in ψ and $\bar{\sigma}_\zeta^2$. ω_0 is

increasing in ϕ and σ_D^2, and so it is $\bar{p}_{ss}$. □

Proof Proposition 6

Proof. Let Q_x^u, Q_ζ^u, X_t^u and Q_x^o, Q_ζ^o, X_t^o be the coefficients and holdings in the unobservable and observable case, respectively. Consider the trajectory of $\mathbb{E}[X_t^u - X_t^o]$, which satisfies

$$\frac{d}{dt}\mathbb{E}[(X_t^u - X_t^o)] = (Q_x^u - Q_x^o)(\bar{X}_{ss} - \mathbb{E}[X_t^u]) - Q_x^o\,\mathbb{E}[(X_t^u - X_t^o)] + (Q_\zeta^u - Q_\zeta^0)\,\mathbb{E}[\zeta_t], \tag{A.30}$$

and initial condition $X_0^u - X_0^o = 0$. Given $\zeta_0 = 0$ we have that $\mathbb{E}[\zeta_t] = 0$ so we can ignore the last term. First, show that $Q_x^u < Q_x^o$. Using the solutions for Q_x^u and Q_x^o we get that,

$$Q_x^o - Q_x^u = \frac{2r\left(1 + 2\alpha^2 - \alpha\right) + (1-2\alpha)\left(3\kappa(1-2\alpha) - \sqrt{\eta^2 + 2r\phi(r+3\kappa)\alpha^2}\right)}{4\alpha(1-2\alpha)}.$$

$\alpha < 1/2$ given the hypothesis in the proposition which means that hte previous expression is positive if and only if

$$2r\left(1 - \alpha(1-2\alpha)\right) + (1-2\alpha)\left(3\kappa(1-2\alpha) - \sqrt{\eta^2 + 2r\phi(r+3\kappa)\alpha^2}\right) > 0.$$

Given $\alpha \in (0, 1/2)$, this conditions is satisfied if and only if

$$\phi < \frac{1}{2}\left(1 + \frac{r}{r+3\kappa}\frac{1 + 2\alpha(1-2\alpha)}{(1-2\alpha)^2}\right)$$

From here we get that $Q_x^o > Q_x^u$ if and only if $\kappa < \kappa^\dagger$ where

$$\kappa^\dagger = \begin{cases} \infty & \text{if } \phi \leq 1/2 \\ \frac{r}{3}\left(\frac{1+2\alpha(1-2\alpha)}{(2\phi-1)(1-2\alpha)^2} - 1\right) & \text{if } \phi > 1/2 \end{cases} \tag{A.31}$$

If $\kappa < \kappa^\dagger$, then we get that $Q_x^o > Q_x^u$ so $(Q_x^u - Q_x^o)(\bar{X} - X_t) > 0$ if and only if $\bar{X} < X_t$. The conclusion follows directly from looking at the trajectories of the ODE in (A.30). A similar analysis follows when $\kappa > \kappa^\dagger$ with the reversed inequalities. □

Proof Proposition 7

Proof. The first step is to verify that $Q_x^o > Q_x^u$. In the limit when σ_ζ^2 and κ go to zero, Proposition 1 implies that

$$Q_x^o = \frac{2r\alpha}{1-2\alpha},$$

where $\alpha \equiv r\phi(\gamma_L + \gamma_M)\sigma_D^2$, and $\alpha < 1/2$ from the hypothesis in the Proposition. On the other hand,

$$Q_x^u = r\frac{\sqrt{8\alpha^2 + (\alpha+1)^2} - \alpha - 1}{4\alpha}.$$

Combining both expressions we get

$$Q_x^o - Q_x^u = r\frac{6\alpha^2 - \alpha + 1 - (1-2\alpha)\sqrt{8\alpha^2 + (\alpha+1)^2}}{4\alpha(1-2\alpha)},$$

which is positive for all $\alpha \in [0, 1/2)$. The next step is to compare Q_0^o and Q_0^u. First, we have from Proposition 1 that

$$Q_0^o = \frac{2r^2\phi\gamma_M\sigma_D^2}{1-2\alpha}.$$

On the other hand, we have

$$Q_0^u = \frac{r\gamma_M\left[\left(\sqrt{8\alpha^2 + (\alpha+1)^2} + 1 - \alpha\right)\theta + \left(\sqrt{8\alpha^2 + (\alpha+1)^2} - (\alpha+1)\right)\right]}{4\alpha\left(\gamma_L + \gamma_M\right)(\theta+1)},$$

where

$$\theta \equiv \frac{\psi^2\gamma_L\bar{\sigma}_\zeta^2}{\phi}.$$

Combining both expressions we get

$$Q_0^o - Q_0^u = r\gamma_M\frac{2(1-2\alpha) + (\theta+1)\left[6\alpha^2 + 3\alpha - 1 - (1-2\alpha)\sqrt{8\alpha^2 + (\alpha+1)^2}\right]}{4\alpha(1-2\alpha)(\theta+1)\left(\gamma_L + \gamma_M\right)}. \tag{A.32}$$

Term $6\alpha^2 + 3\alpha - 1 - (1-2\alpha)\sqrt{8\alpha^2 + (\alpha+1)^2}$ is negative if $\alpha < \frac{1}{16}\left(1 + \sqrt{17}\right)$ and positive if $\alpha > \frac{1}{16}\left(1 + \sqrt{17}\right)$. Thus, if $\frac{1}{16}\left(1 + \sqrt{17}\right) < \alpha < \frac{1}{2}$, then the expression in (A.32) is always positive. On the other hand, if $\alpha < \frac{1}{16}\left(1 + \sqrt{17}\right)$, then the expression in (A.32) is positive if and only if

$$\theta < \frac{(1-2\alpha)\left(1 + \alpha - \sqrt{8\alpha^2 + (\alpha+1)^2}\right) + 8\alpha^2}{(1-2\alpha)\left(1 - \alpha + \sqrt{8\alpha^2 + (\alpha+1)^2}\right) - 8\alpha^2}.$$

From here we get that $Q_0^o > Q_0^u$ if and only if $\psi^2\bar{\sigma}_\zeta^2 < z^\dagger \equiv \phi k(\alpha)/\gamma_L$ where

$$k(\alpha) = \begin{cases} \infty & \text{if } \frac{1}{16}\left(1+\sqrt{17}\right) \geq \alpha \\ \frac{(1-2\alpha)\left(1+\alpha-\sqrt{8\alpha^2+(\alpha+1)^2}\right)+8\alpha^2}{(1-2\alpha)\left(1-\alpha+\sqrt{8\alpha^2+(\alpha+1)^2}\right)-8\alpha^2} & \text{if } \frac{1}{16}\left(1+\sqrt{17}\right) < \alpha \end{cases} \tag{A.33}$$

Next, we look at the ODE for the

$$\frac{d}{dt}\,\mathbb{E}[(X_t^u - X_t^o)] = Q_0^u - Q_0^o + (Q_x^o - Q_x^u)\,\mathbb{E}[X_t^u] - Q_x^o\,\mathbb{E}[(X_t^u - X_t^o)]. \tag{A.34}$$

with initial condition $X_0^o - X_0^u = 0$. We need then to consider two cases: (1) $Q_0^u > Q_0^o$ and (2) $Q_0^o > Q_0^u$.

Case (1): $\psi^2\bar{\sigma}_\zeta^2 \geq z^\dagger$ so $Q_0^u \geq Q_0^0$ is always positive. If $Q_0^u \geq Q_0^o$ and $\mathbb{E}[X_t^u] \geq 0$ then $\mathbb{E}[(X_t^u - X_t^o)] = 0$ implies $\frac{d}{dt}\,\mathbb{E}[(X_t^u - X_t^o)] \geq 0$, which means that $\mathbb{E}[(X_t^u - X_t^o)] \geq 0$ for $t > 0$. Moreover, looking at the second derivative $\frac{d^2}{dt^2}\,\mathbb{E}[(X_t^u - X_t^o)]$ we can verify that the weak inequality actually is strict.

Case (2): $\psi^2\bar{\sigma}_\zeta^2 < z^\dagger$ so $Q_0^u < Q_x^u$. In this case we have that

$$\frac{d}{dt}\,\mathbb{E}[(X_t^u - X_t^o)]\Big|_{\mathbb{E}[(X_t^u - X_t^o)]=0} > 0 \Leftrightarrow \mathbb{E}[X_t^u] > x_0^\dagger \equiv \frac{Q_0^o - Q_0^u}{Q_x^o - Q_x^u}$$

where

$$\frac{Q_0^o - Q_0^u}{Q_x^o - Q_x^u} = \frac{Q_x^o\bar{X}_{ss}^o - Q_x^u\bar{X}_{ss}^u}{Q_x^o - Q_x^u} < \frac{Q_x^o\bar{X}_{ss}^o - Q_x^u\bar{X}_{ss}^o}{Q_x^o - Q_x^u} = \bar{X}_{ss}^o.$$

Substituting $Q_0^o - Q_0^u$ and $Q_x^o - Q_x^u$ we get that

$$x_0^\dagger = \frac{\gamma_M}{\gamma_L + \gamma_M}\left(\frac{2(1-2\alpha) + (\theta+1)\left[6\alpha^2 + 3\alpha - 1 - (1-2\alpha)\sqrt{8\alpha^2+(\alpha+1)^2}\right]}{(\theta+1)\left[6\alpha^2 - \alpha + 1 - (1-2\alpha)\sqrt{8\alpha^2+(\alpha+1)^2}\right]}\right)$$

From here we get that if $X_0^o = X_0^u > x_0^\dagger$ then $\mathbb{E}[X_t^u] - \mathbb{E}[X_t^o] > 0$ for all $t > 0$. If $X_0^o = X_0^u = x_0(\alpha,\theta)$ we can verify that $\frac{d^2}{dt^2}\,\mathbb{E}[(X_t^u - X_t^o)]\Big|_{t=0} > 0$ so by it also follows that $\mathbb{E}[X_t^u] - \mathbb{E}[X_t^o] > 0$ for all $t > 0$. On the other hand, if $X_0^o = X_0^u < x_0^\dagger$, then $\mathbb{E}[(X_t^u - X_t^o)]$ single crosses zero from below, which means that there is t^* such that $\mathbb{E}[X_t^u] - \mathbb{E}[X_t^o] < 0$ on $(0, t^*)$ and $\mathbb{E}[X_t^u] - \mathbb{E}[X_t^o] > 0$ on (t^*, ∞). □

B Model with Liquidity Shocks

Proof Proposition 3

Proof. Using the definition of I_t in (15), we get that

$$\begin{aligned} dI_t &= d\zeta_t + \frac{Q_b}{Q_\zeta} db_t \\ &= -\kappa\zeta_t dt - \lambda\frac{Q_b}{Q_\zeta} b_t + \sigma_\zeta dB_t^\zeta + \frac{Q_b}{Q_\zeta}\sigma_b dB_t^b \\ &= -\kappa\left(I_t - \frac{Q_b}{Q_\zeta} b_t\right) dt - \lambda\frac{Q_b}{Q_\zeta} b_t + \sigma_\zeta dB_t^\zeta + \frac{Q_b}{Q_\zeta}\sigma_b dB_t^b \\ &= -\kappa I_t dt + (\kappa - \lambda)\frac{Q_b}{Q_\zeta} b_t + \sigma_\zeta dB_t^\zeta + \frac{Q_b}{Q_\zeta}\sigma_b dB_t^b, \end{aligned}$$

where in the third line we have use the relation

$$I_t = \zeta_t + \frac{Q_b}{Q_\zeta} b_t.$$

On the other hand, given the conjectured equilibrium effort and the definition of I_t, we can write the stochastic differential equations for the cumulative dividends process as

$$\begin{aligned} dD_t &= (\mu_D + A_0 + A_x X_t + A_\zeta \zeta_t + A_b b_t + A_{\hat{b}}\hat{b}_t)dt + \sigma_D dB_t^D \\ &= \left(\mu_D + A_0 + A_x X_t + A_\zeta I_t - A_\zeta \frac{Q_b}{Q_\zeta} b_t + A_b b_t + A_{\hat{b}}\hat{b}_t\right) dt + \sigma_D dB_t^D. \end{aligned}$$

From here, we get a standard single dimensional filtering problem for b_t with the observation process

$$\begin{aligned} dI_t &= \left(-\kappa I_t + (\kappa - \lambda)\frac{Q_b}{Q_\zeta} b_t\right) dt + \sigma_\zeta dB_t^\zeta + \frac{Q_b}{Q_\zeta}\sigma_b dB_t^b \\ dD_t &= \left(\mu_D + A_0 + A_x X_t + A_\zeta I_t - A_\zeta \frac{Q_b}{Q_\zeta} b_t + A_b b_t + A_{\hat{b}}\hat{b}_t\right) dt + \sigma_D dB_t^D \end{aligned}$$

Adapting the notation in Liptser and Shiryaev (2001b) to our problem we get

$$
\begin{aligned}
a_0(t) &= 0 \\
a_1(t) &= -\lambda \\
b_1(t) &= \sigma_b \\
b_2(t) &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\
A_0(t) &= \begin{pmatrix} -\kappa I_t \\ \mu_D + A_0 + A_x X_t + A_{\hat{b}} \hat{b}_t + A_\zeta I_t \end{pmatrix} \\
A_1(t) &= \begin{pmatrix} (\kappa - \lambda)\frac{Q_b}{Q_\zeta} \\ A_b - A_\zeta \frac{Q_b}{Q_\zeta} \end{pmatrix} \\
B_1(t) &= \begin{pmatrix} \frac{Q_b}{Q_\zeta}\sigma_b \\ 0 \end{pmatrix} \\
B_2(t) &= \begin{pmatrix} \sigma_\zeta & 0 \\ 0 & \sigma_D \end{pmatrix}
\end{aligned}
$$

Using Theorem 12.7 in Lipster and Shiryaev we get

$$
\begin{aligned}
d\hat{b}_t = -\lambda \hat{b}_t dt + \beta_q \left(dI_t + \left(\kappa I_t - (\kappa - \lambda)\frac{Q_b}{Q_\zeta}\hat{b}_t \right) dt \right) \\
+ \beta_D \left(dD_t - \left(\mu_D + A_0 + A_x X_t + A_{\hat{b}}\hat{b}_t + A_\zeta I_t + \left(A_b - A_\zeta \frac{Q_b}{Q_\zeta} \right) \hat{b}_t \right) dt \right)
\end{aligned}
$$

where

$$
\begin{aligned}
\begin{pmatrix} \beta_q & \beta_D \end{pmatrix} &= (b_1 B_1^\intercal + b_2 B_2^\intercal + \sigma_{\hat{b}}^2 A_1^\intercal)(B_1 B_1^\intercal + B_2 B_2^\intercal)^{-1} \\
0 &= 2a_1\sigma_{\hat{b}}^2 + b_1 b_1^\intercal + b_2 b_2^\intercal - (b_1 B_1^\intercal + b_2 B_2^\intercal + \sigma_{\hat{b}}^2 A_1^\intercal)(B_1 B_1^\intercal + B_2 B_2^\intercal)^{-1}(b_1 B_1^\intercal + b_2 B_2^\intercal + \sigma_{\hat{b}}^2 A_1^\intercal)^\intercal.
\end{aligned}
$$

From here we get that (β_q, β_D) is given by

$$
\beta_q = \frac{\sigma_b^2 + (\kappa - \lambda)\sigma_{\hat{b}}^2}{\left(\sigma_\zeta^2 + \frac{Q_b}{Q_\zeta}\right)^2 \sigma_b^2}\frac{Q_b}{Q_\zeta} \tag{B.1}
$$

$$
\beta_D = \frac{\sigma_{\hat{b}}^2}{\sigma_D^2}\left(A_b - A_\zeta \frac{Q_b}{Q_\zeta} \right) \tag{B.2}
$$

and the stationary variance of $\hat{b}_t$, which we denote by $\sigma^2_{\hat{b}}$, is the positive root of the following quadratic equation.

$$0 = -2\lambda\sigma^2_{\hat{b}} + \sigma^2_b - \left[\frac{\left(\sigma^2_b + (\kappa-\lambda)\sigma^2_{\hat{b}}\right)^2}{\sigma^2_\zeta + \left(\frac{Q_b}{Q_\zeta}\right)^2 \sigma^2_b} \left(\frac{Q_b}{Q_\zeta}\right)^2 + \frac{\sigma^4_{\hat{b}}}{\sigma^2_D}\left(A_b - A_\zeta \frac{Q_b}{Q_\zeta}\right)^2 \right] \tag{B.3}$$

Next, we express the stochastic differential equation for $\hat{b}_t$ in term of the innovation processes $\tilde{B}^\zeta_t, \tilde{B}^b_t, \tilde{B}^D_t$. We can write

$$\begin{aligned}
dI_t + \left(\kappa I_t - (\kappa-\lambda)\frac{Q_b}{Q_\zeta}\hat{b}_t\right)dt &= \kappa\frac{Q_b}{Q_\zeta}(b_t - \hat{b}_t)dt + \sigma_\zeta dB^\zeta_t + \frac{Q_b}{Q_\zeta}\left(-\lambda b_t dt + \sigma_b dB^b_t + \lambda\hat{b}_t dt\right) \\
&= \kappa\frac{Q_b}{Q_\zeta}(b_t - \hat{b}_t)dt + \sigma_\zeta dB^\zeta_t + \frac{Q_b}{Q_\zeta}\sigma_b d\tilde{B}^b_t \\
&= \kappa(\hat{\zeta}_t - \zeta_t)dt + \sigma_\zeta dB^\zeta_t + \frac{Q_b}{Q_\zeta}\sigma_b d\tilde{B}^b_t \\
&= \sigma_\zeta d\tilde{B}^\zeta_t + \frac{Q_b}{Q_\zeta}\sigma_b d\tilde{B}^b_t
\end{aligned}$$

where we have used the relation

$$I_t = \zeta_t + \frac{Q_b}{Q_\zeta}b_t = \hat{\zeta}_t + \frac{Q_b}{Q_\zeta}\hat{b}_t.$$

Moreover, the previous relation also implies that $\hat{\zeta}_t$ and ζ_t are related as follows

$$\hat{\zeta}_t - \zeta_t = \frac{Q_b}{Q_\zeta}(b_t - \hat{b}_t).$$

Hence, we arrive to the SDE for $\hat{b}_t$ in the proposition

$$d\hat{b}_t = -\lambda\hat{b}_t dt + \beta_q\left(\sigma_\zeta d\tilde{B}^\zeta_t + \frac{Q_b}{Q_\zeta}\sigma_b d\tilde{B}^b_t\right) + \beta_D\sigma_D d\tilde{B}^D_t.$$

The final step is to find the SDE for $\hat{\zeta}_t$. By definition, $d\hat{\zeta}_t = dI_t - \frac{Q_b}{Q_\zeta} d\hat{b}_t$, hence we can write

$$\begin{aligned}
d\hat{\zeta}_t &= -\kappa I_t dt + (\kappa - \lambda)\frac{Q_b}{Q_\zeta} b_t + \sigma_\zeta dB_t^\zeta + \frac{Q_b}{Q_\zeta}\sigma_b dB_t^b - \frac{Q_b}{Q_\zeta} d\hat{b}_t \\
&= -\kappa\left(\zeta_t + \frac{Q_b}{Q_\zeta} b_t\right) dt + (\kappa - \lambda)\frac{Q_b}{Q_\zeta} b_t + \sigma_\zeta dB_t^\zeta + \frac{Q_b}{Q_\zeta}\sigma_b dB_t^b - \frac{Q_b}{Q_\zeta} d\hat{b}_t \\
&= -\kappa\hat{\zeta}_t dt + \sigma_\zeta\left(1 - \frac{Q_b}{Q_\zeta}\beta_q\right) d\tilde{B}_t^\zeta + \sigma_b \frac{Q_b}{Q_\zeta}\left(1 - \beta_q \frac{Q_b}{Q_\zeta}\right) d\tilde{B}_t^b - \sigma_D \beta_D \frac{Q_b}{Q_\zeta} d\tilde{B}_t^D,
\end{aligned}$$

which corresponds to expression in the Proposition. Finally, by the innovation theorem (Liptser and Shiryaev, 2001a, Theorem 7.17), the processes $\tilde{B}_t^\zeta, \tilde{B}_t^b, \tilde{B}_t^D$ are standard Brownian motions under $\mathcal{F}_t^{q,D}$ □

Proof Lemma 4

Proof. We derive the stochastic differential equation for $\hat{b}_t$ given the blockholder's filtration $\mathcal{F}_t^{D,b,\zeta}$. Given an arbitrary strategy $\tilde{a}_t$ and $\tilde{q}_t$ we have that

$$\begin{aligned}
d\hat{b}_t = -\lambda\hat{b}_t dt - \beta_D\left(A_0 + A_x X_t + A_{\hat{b}}\hat{b}_t + A_\zeta I_t + \left(A_b - A_\zeta\frac{Q_b}{Q_\zeta}\right)\hat{b}_t - \tilde{a}_t\right) dt + \beta_D\sigma_D dB_t^D \\
+ \beta_q\left(dI_t + \left(\kappa I_t - (\kappa - \lambda)\frac{Q_b}{Q_\zeta}\hat{b}_t\right) dt\right)
\end{aligned}$$

Substituting I_t in equation (16) we get

$$\begin{aligned}
d\hat{b}_t = -\lambda\hat{b}_t dt - \beta_D\left(A_0 + A_x X_t + A_{\hat{b}}\hat{b}_t + A_\zeta\left(\frac{\tilde{q}_t - Q_0 + Q_x X_t - Q_{\hat{b}}\hat{b}_t}{Q_\zeta}\right) + \left(A_b - A_\zeta\frac{Q_b}{Q_\zeta}\right)\hat{b}_t - \tilde{a}_t\right) dt \\
+ \beta_D\sigma_D dB_t^D + \frac{\beta_q}{Q_\zeta}\left(d\tilde{q}_t + \left(\kappa\left(\tilde{q}_t - Q_0 + Q_x X_t - Q_{\hat{b}}\hat{b}_t\right) - (\kappa - \lambda)Q_b\hat{b}_t\right) dt\right)
\end{aligned}$$

Letting $\Delta_t = \tilde{q}_t - q_t$ we get

$$\begin{aligned}
d\hat{b}_t = -\lambda\hat{b}_t dt - \beta_D\left(A_0 + A_x X_t + A_{\hat{b}}\hat{b}_t + A_\zeta\left(\zeta_t + \frac{Q_b}{Q_\zeta} b_t\right) + \left(A_b - A_\zeta\frac{Q_b}{Q_\zeta}\right)\hat{b}_t - \tilde{a}_t\right) dt \\
+ \left(\beta_q\frac{\kappa}{Q_\zeta} - \beta_D\frac{A_\zeta}{Q_\zeta}\right)\Delta_t dt + \frac{\beta_q}{Q_\zeta} d\Delta_t + \beta_q\sigma_\zeta dB_t^\zeta + \beta_q\frac{Q_b}{Q_\zeta}\sigma_b dB_t^b + \beta_D\sigma_D dB_t^D
\end{aligned}$$

so we get

$$d\hat{b}_t = \mu_{\hat{b}}(X_t, b_t, \hat{b}_t, \zeta_t, \Delta_t)dt + \beta_D \tilde{a}_t dt + \frac{\beta_q}{Q_\zeta} d\Delta_t + \beta_q \sigma_\zeta dB_t^\zeta + \beta_q \frac{Q_b}{Q_\zeta} \sigma_b dB_t^b + \beta_D \sigma_D dB_t^D$$

where

$$\mu_{\hat{b}}(X_t, b_t, \hat{b}_t, \zeta_t, \Delta_t) = \mathcal{B}_0 + \mathcal{B}_x X_t + \mathcal{B}_b b_t - \mathcal{B}_{\hat{b}} \hat{b}_t + \mathcal{B}_\zeta \zeta_t + \mathcal{B}_\Delta \Delta_t$$

and

$$\mathcal{B}_0 = -\beta_D A_0 \qquad \mathcal{B}_x = -\beta_D A_x \tag{B.4}$$

$$\mathcal{B}_b = -\beta_D A_\zeta \frac{Q_b}{Q_\zeta} \qquad \mathcal{B}_{\hat{b}} = \lambda + \beta_D \left(A_b + A_{\hat{b}} - A_\zeta \frac{Q_b}{Q_\zeta} \right) \tag{B.5}$$

$$\mathcal{B}_\zeta = -\beta_D A_\zeta \qquad \mathcal{B}_\Delta = \beta_q \frac{\kappa}{Q_\zeta} - \beta_D \frac{A_\zeta}{Q_\zeta} \tag{B.6}$$

□

Proof Lemma 8

Proof. The HJB equation for the competitive investor optimization problem is

$$\begin{aligned} rJ = \max_{c,q} u_M(c) + (rW - c - pq + (\mu_D + A_0 + A_x X + A_\zeta \hat{\zeta}_t + (A_b + A_{\hat{b}})\hat{b}_t)Y) J_W - \kappa \hat{\zeta} J_{\hat{\zeta}} - \lambda \hat{b} J_{\hat{b}} \\ + \left(Q_0 - Q_x X + Q_\zeta \hat{\zeta} + \left(Q_b + Q_{\hat{b}} \right) \hat{b} \right) J_x + q J_y + \frac{1}{2} \left[\sigma_D^2 Y^2 J_{WW} + \left(\beta_q^2 \sigma_\zeta^2 + \sigma_b^2 \beta_q^2 \left(\frac{Q_b}{Q_\zeta} \right)^2 + \beta_D^2 \sigma_D^2 \right) J_{\hat{b}\hat{b}} \right. \\ + \left(\sigma_\zeta^2 \left(1 - \frac{Q_b}{Q_\zeta} \beta_q \right)^2 + \sigma_b^2 \left(\frac{Q_b}{Q_\zeta} \right)^2 \left(1 - \beta_q \frac{Q_b}{Q_\zeta} \right)^2 + \sigma_D^2 \beta_D^2 \left(\frac{Q_b}{Q_\zeta} \right)^2 \right) J_{\hat{\zeta}\hat{\zeta}} \\ \left. + 2\sigma_D^2 \beta_D Y J_{W\hat{b}} - 2\sigma_D^2 \beta_D Y \frac{Q_b}{Q_\zeta} J_{W\hat{\zeta}} + \left(\sigma_\zeta^2 \beta_q \left(1 - \frac{Q_b}{Q_\zeta} \beta_q \right) + \sigma_b^2 \beta_q \frac{Q_b}{Q_\zeta} \left(1 - \beta_q \frac{Q_b}{Q_\zeta} \right) - \sigma_D^2 \beta_D^2 \frac{Q_b}{Q_\zeta} \right) J_{\hat{b}\hat{\zeta}} \right] \end{aligned}$$

As we did in the model without liquidity shocks, we conjecture a value function

$$J(W, Y, X, \hat{b}, \hat{\zeta}) = -\frac{\exp\left(-r\gamma_M \left(W_M + H(Y, X, \hat{b}, \hat{\zeta}) \right) \right)}{r}$$

The first order condition for consumption is

$$u'_M(c) = J_W,$$

so

$$u_M(c) = rJ$$

and

$$c = rW_M + rH(Y, X, \hat{b}, \hat{\zeta})$$

Substituting our conjecture for the value function and the first order condition for consumption, and defining

$$\Sigma_{\hat{b}} \equiv \beta_q^2 \sigma_\zeta^2 + \sigma_b^2 \beta_q^2 \left(\frac{Q_b}{Q_\zeta}\right)^2 + \beta_D^2 \sigma_D^2 \tag{B.7}$$

$$\Sigma_{\hat{z}} \equiv \sigma_\zeta^2 \left(1 - \frac{Q_b}{Q_\zeta}\beta_q\right)^2 + \sigma_b^2 \left(\frac{Q_b}{Q_\zeta}\right)^2 \left(1 - \beta_q \frac{Q_b}{Q_\zeta}\right)^2 + \sigma_D^2 \beta_D^2 \left(\frac{Q_b}{Q_\zeta}\right)^2 \tag{B.8}$$

$$\Sigma_{\hat{b}\hat{z}} \equiv \sigma_\zeta^2 \beta_q \left(1 - \frac{Q_b}{Q_\zeta}\beta_q\right) + \sigma_b^2 \beta_q \frac{Q_b}{Q_\zeta}\left(1 - \beta_q \frac{Q_b}{Q_\zeta}\right) - \sigma_D^2 \beta_D^2 \frac{Q_b}{Q_\zeta} \tag{B.9}$$

we get

$$\begin{aligned} rH = \max_q (\mu_D + A_0 + A_x X + A_\zeta \hat{\zeta}_t + (A_b + A_{\hat{b}})\hat{b})Y - pq - \frac{r\gamma_M}{2}\Big[\sigma_D^2 Y^2 + 2\sigma_D^2 \beta_D \left(H_{\hat{b}} - \frac{Q_b}{Q_\zeta} H_{\hat{\zeta}}\right) Y \\ + \Sigma_{\hat{b}} H_{\hat{b}}^2 + \Sigma_{\hat{\zeta}} H_{\hat{\zeta}}^2 + 2\Sigma_{\hat{b}\hat{\zeta}} H_{\hat{b}} H_{\hat{\zeta}}\Big] + \left(Q_0 - Q_x X + Q_\zeta \hat{\zeta} + \left(Q_b + Q_{\hat{b}}\right)\hat{b}\right) H_x \\ + q H_y - \kappa \hat{\zeta} H_{\hat{\zeta}} - \lambda \hat{b} H_{\hat{b}} + \frac{1}{2}\left[\Sigma_{\hat{b}} H_{\hat{b}\hat{b}} + \Sigma_{\hat{\zeta}} H_{\hat{\zeta}\hat{\zeta}} + 2\Sigma_{\hat{b}\hat{\zeta}} H_{\hat{b}\hat{\zeta}}\right]. \end{aligned}$$

Using the first conditions for q, i.e. $p = H_y$, we get

$$\begin{aligned} rH = (\mu_D + A_0 + A_x X + A_\zeta \hat{\zeta} + (A_b + A_{\hat{b}})\hat{b})Y - \frac{r\gamma_M}{2}\Big[\sigma_D^2 Y^2 + 2\sigma_D^2 \beta_D \left(H_{\hat{b}} - \frac{Q_b}{Q_\zeta} H_{\hat{\zeta}}\right) Y \\ + \Sigma_{\hat{b}} H_{\hat{b}}^2 + \Sigma_{\hat{\zeta}} H_{\hat{\zeta}}^2 + 2\Sigma_{\hat{b}\hat{\zeta}} H_{\hat{b}} H_{\hat{\zeta}}\Big] + \left(Q_0 - Q_x X + Q_\zeta \hat{\zeta} + \left(Q_b + Q_{\hat{b}}\right)\hat{b}\right) H_x \\ - \kappa \hat{\zeta} H_{\hat{\zeta}} - \lambda \hat{b} H_{\hat{b}} + \frac{1}{2}\left[\Sigma_{\hat{b}} H_{\hat{b}\hat{b}} + \Sigma_{\hat{\zeta}} H_{\hat{\zeta}\hat{\zeta}} + 2\Sigma_{\hat{b}\hat{\zeta}} H_{\hat{b}\hat{\zeta}}\right]. \end{aligned}$$

□

Proof of Proposition 9

Proof. As in the proof in Lemma 8, if G satisfies

$$\begin{aligned} rG = \max_{a} \ & (\mu_D + a - \delta b)X - R(X, \hat{b}, q^L + \Delta)(q^L + \Delta) - \Phi(a, \zeta) \\ & - \frac{r\gamma_L}{2}\left[\sigma_D^2 X^2 + 2\sigma_D^2 \beta_D G_{\hat{b}} X + \Sigma_{\hat{b}} G_{\hat{b}}^2 + \sigma_b^2 G_b^2 + \sigma_\zeta G_\zeta^2 + \beta_q \sigma_\zeta^2 G_{\hat{b}} G_\zeta + \beta_q \frac{Q_b}{Q_\zeta}\sigma_b^2 G_{\hat{b}} G_b\right] \\ & - \kappa\zeta G_\zeta - \lambda b G_b + \left(\mu_{\hat{b}}(X, b, \hat{b}, \zeta, \Delta) + \beta_D a\right) G_{\hat{b}} + (q^L + \Delta) G_X \\ & + \frac{1}{2}\left[\Sigma_{\hat{b}} G_{\hat{b}\hat{b}} + \sigma_b^2 G_{bb} + \sigma_\zeta G_{\zeta\zeta} + \beta_q \sigma_\zeta^2 G_{\hat{b}\zeta} + \beta_q \frac{Q_b}{Q_\zeta}\sigma_b^2 G_{\hat{b}b}\right], \end{aligned}$$

then the function

$$V(W, X, \zeta, b, \hat{b}, \Delta) = -\frac{\exp\left(-r\gamma_L\left(W + G(X, \zeta, b, \hat{b}, \Delta)\right)\right)}{r}$$

satisfies the HJB equation

$$\begin{aligned} rV = \max_{c,a} \ & u_L(c) + (rW - c - R(q^L + \Delta, X, \hat{b})(q^L + \Delta) - \Phi(a, \zeta) + (\mu_D + a - \delta b)X)V_W \\ & - \kappa\zeta V_\zeta - \lambda b V_b + \left(\mu_{\hat{b}}(X, b, \hat{b}, \zeta, \Delta) + \beta_D a\right) V_{\hat{b}} + q^L V_x \\ & + \frac{1}{2}\left[\sigma_D^2 X^2 V_{WW} + \Sigma_{\hat{b}} V_{\hat{b}\hat{b}} + \sigma_b^2 V_{bb} + \sigma_\zeta V_{\zeta\zeta} + 2\sigma_D^2 \beta_D X V_{W\hat{b}} + \beta_q \sigma_\zeta^2 V_{\hat{b}\zeta} + \beta_q \frac{Q_b}{Q_\zeta}\sigma_b^2 V_{\hat{b}b}\right]. \end{aligned}$$

To simplify the notation, let's define the infinitesimal operator given a policy $(a_t, c_t)_{t\geq 0}$ as

$$\begin{aligned} \mathcal{D}^{a,c} f \equiv & \left(rW - c - R(q^L + \Delta, X, \hat{b})(q + \Delta) - \Phi(a, \zeta) + (\mu_D + a - \delta b)X\right) f_W \\ & - \kappa\zeta f_\zeta - \lambda b V_b + \left(\mu_{\hat{b}}(X, b, \hat{b}, \zeta, \Delta) + \beta_D a\right) f_{\hat{b}} + (q^L + \Delta) f_x + \dot{\Delta} f_\Delta \\ & + \frac{1}{2}\left[\sigma_D^2 X^2 f_{WW} + \Sigma_{\hat{b}} f_{\hat{b}\hat{b}} + \sigma_b^2 f_{bb} + \sigma_\zeta f_{\zeta\zeta} + 2\sigma_D^2 \beta_D X f_{W\hat{b}} + \beta_q \sigma_\zeta^2 f_{\hat{b}\zeta} + \beta_q \frac{Q_b}{Q_\zeta}\sigma_b^2 f_{\hat{b}b}\right]. \end{aligned}$$

Consider an arbitrary policy $(\tilde{c}_t, \tilde{a}_t, \tilde{q}_t)$, and apply Itô's Lemma to $e^{-rt}V(W_t, X_t, \zeta_t, b_t, \hat{b}_t, \Delta_t)$ to get

$$\begin{aligned}
\mathbb{E}\left[e^{-rt}V(W_t, X_t, \zeta_t, b_t, \hat{b}_t, \Delta_t)\right] &= V(W_0, X_0, \zeta_0, b_0, \hat{b}_0, \Delta_0) \\
&\quad + \mathbb{E}\left[\int_0^t e^{-rs}\left(\mathcal{D}^{\tilde{a},\tilde{c}}V(W_s, X_s, \zeta_s, b_s, \hat{b}_s, \Delta_s) - rV(W_s, X_s, \zeta_s, b_s, \hat{b}_s, \Delta_s)\right)ds\right] \\
&\quad + \mathbb{E}\left[\int_0^t e^{-rs}\left(V_\Delta(X_s, \zeta_s, b_s, \hat{b}_s, \Delta_s) + \frac{\beta_q}{Q_\zeta}V_{\hat{b}}(X_s, \zeta_s, b_s, \hat{b}_s, \Delta_s)\right)\dot{\Delta}_s ds\right] \\
&\leq V(W_0, X_0, \zeta_0, b_0, \hat{b}_0, \Delta_0) - \mathbb{E}\left[\int_0^t e^{-rs}u(\tilde{c}_s)ds\right] \qquad \text{(B.10)} \\
&\quad + \mathbb{E}\left[\int_0^t e^{-rs}\left(V_\Delta(X_s, \zeta_s, b_s, \hat{b}_s, \Delta_s) + \frac{\beta_q}{Q_\zeta}V_{\hat{b}}(X_s, \zeta_s, b_s, \hat{b}_s, \Delta_s)\right)\dot{\Delta}_s ds\right],
\end{aligned}$$

where the inequality follows from $u(\tilde{c}) + \mathcal{D}^{\tilde{a},\tilde{c}}V \leq rV$. To keep the expression that follow short, for any function f, we let $f(s) \equiv f(X_s, \zeta_s, b_s, \hat{b}_s, \Delta_s)$. Because V is an exponential of the function $G(\cdot)$, we have that

$$\mathbb{E}\left[\int_0^t e^{-rs}\left(V_\Delta(s) + \frac{\beta_q}{Q_\zeta}V_{\hat{b}}(s)\right)\dot{\Delta}_s ds\right] = r\gamma_L\,\mathbb{E}\left[\int_0^t e^{-rs}V(s)\Big(G_\Delta(s) + \frac{\beta_q}{Q_\zeta}G_{\hat{b}}(s)\Big)\dot{\Delta}_s ds\right].$$

Using the integration by parts formula for semimartingales (Karatzas and Shreve, 2012), we get

$$\begin{aligned}
\mathbb{E}\left[\int_0^t e^{-rs}V(s)\Big(G_\Delta(s) + \frac{\beta_q}{Q_\zeta}G_{\hat{b}}(s)\Big)\dot{\Delta}_s ds\right] &= \mathbb{E}\left[e^{-rt}V(t)\Big(G_\Delta(t) + \frac{\beta_q}{Q_\zeta}G_{\hat{b}}(t)\Big)\Delta_t\right] \\
&\quad - \mathbb{E}\left[\int_0^t e^{-rs}(\mathcal{D}^{\tilde{a},\tilde{c}}V(s) - rV(s))\Big(G_\Delta(s) + \frac{\beta_q}{Q_\zeta}G_{\hat{b}}(s)\Big)\Delta_s ds\right. \\
&\quad \left. + \int_0^t e^{-rs}V(s)\Delta_s\mathcal{D}^{\tilde{a},\tilde{c}}\Big(G_\Delta(s) + \frac{\beta_q}{Q_\zeta}G_{\hat{b}}(s)\Big)ds\right]. \qquad \text{(B.11)}
\end{aligned}$$

Using the fact that G is linear quadratic, together with the local IC constraint (22a), we get

$$G_\Delta(s) + \frac{\beta_q}{Q_\zeta}G_{\hat{b}}(s) = 2g_{\Delta\Delta}\Delta_s + \frac{\beta_q}{Q_\zeta}g_{\Delta\hat{b}}\Delta_s,$$

which means that

$$\Delta_s\mathcal{D}^{a,c}\left(G_\Delta(s) + \frac{\beta_q}{Q_\zeta}G_{\hat{b}}(s)\right) = \left(G_\Delta(s) + \frac{\beta_q}{Q_\zeta}G_{\hat{b}}(s)\right)\dot{\Delta}_s. \qquad \text{(B.12)}$$

Substituting (B.12) in (B.11), we get

$$\mathbb{E}\left[\int_0^t e^{-rs}V(s)\Big(G_\Delta(s)+\frac{\beta_q}{Q_\zeta}G_{\hat{b}}(s)\Big)\dot{\Delta}_s ds\right] = \frac{1}{2}\,\mathbb{E}\left[e^{-rt}V(t)\Big(G_\Delta(t)+\frac{\beta_q}{Q_\zeta}G_{\hat{b}}(t)\Big)\Delta_t\right] - \frac{1}{2}\,\mathbb{E}\left[\int_0^t e^{-rs}(\mathcal{D}^{a,c}V(s)-rV(s))\Big(G_\Delta(s)+\frac{\beta_q}{Q_\zeta}G_{\hat{b}}(s)\Big)\Delta_s ds\right].$$

Equation (22b) together with the HJB equation for V imply that for any policy the following inequality is satisfied

$$\left(\mathcal{D}^{\tilde{a},\tilde{c}}V(s)-rV(s)\right)\Big(G_\Delta(s)+\frac{\beta_q}{Q_\zeta}G_{\hat{b}}(s)\Big)\Delta_s \geq 0. \tag{B.13}$$

Substituting (B.13) in (B.10), we arrive to

$$\begin{aligned} V(0) &\geq \mathbb{E}\left[\int_0^t e^{-rs}u(\tilde{c}_s)ds\right] - \frac{r\gamma_L}{2}\,\mathbb{E}\left[e^{-rt}V(t)\Big(G_\Delta(t)+\frac{\beta_q}{Q_\zeta}G_{\hat{b}}(t)\Big)\Delta_t\right] \\ &\quad + \mathbb{E}\left[\int_0^t e^{-rs}(\mathcal{D}^{a,c}V(s)-rV(s))\Big(G_\Delta(s)+\frac{\beta_q}{Q_\zeta}G_{\hat{b}}(s)\Big)\Delta_s ds\right] + \mathbb{E}\left[e^{-rt}V(t)\right] \\ &\geq \mathbb{E}\left[\int_0^t e^{-rs}u(\tilde{c}_s)ds\right] - \frac{r\gamma_L}{2}\,\mathbb{E}\left[e^{-rt}V(t)\Big(G_\Delta(t)+\frac{\beta_q}{Q_\zeta}G_{\hat{b}}(t)\Big)\Delta_t\right] + \mathbb{E}\left[e^{-rt}V(t)\right] \end{aligned}$$

Taking the limit when $t\to\infty$, and using the transversality condition, we get that

$$V(W_0,X_0,\zeta_0,b_0,\hat{b}_0,\Delta_0) \geq \mathbb{E}\left[\int_0^\infty e^{-rs}u(\tilde{c}_s)ds\right].$$

In particular, at any time t, $V(W_t,X_t,\zeta_t,b_t,\hat{b}_t,0)$ provides an upper bound for the payoff that the blockholder can get by deviating from the equilibrium strategy $(c_s^L,a_s^L,q_s^L)_{s\geq t}$ from time t onward. Finally, in the case of the strategy (c_t^L,a_t^L,q_t^L), all the inequalities hold with equality so

$$V(W_0,X_0,\zeta_0,b_0,\hat{b}_0,0) = \mathbb{E}\left[\int_0^\infty e^{-rs}u(c_s^L)ds\right],$$

which establishes the optimality of (c_t^L,a_t^L,q_t^L). □

Proof Proposition 10

Proof. On the equilibrium path, the solution for the effort strategy is

$$a_t = \frac{\psi\zeta_t + X_t + \beta_D G_{\hat{b}}(X_t, \zeta_t, b_t, \hat{b}_t, 0)}{2\phi} \tag{B.14}$$

We have shown that the value function is given by

$$V(W, X, \zeta, b, \hat{b}, \Delta) = -\frac{\exp\left(-r\gamma_L\left(W + G(X, \zeta, b, \hat{b}, \Delta)\right)\right)}{r}$$

so we have that

$$G_{\hat{b}} = \frac{V_{\hat{b}}}{V_W}$$

Moreover, the value function V satisfies the HJB equation

$$\begin{aligned} rV = \max_{c,a,\dot{\Delta}}\ & u_L(c) + (rW - c - R(X, \hat{b}, q+\Delta)(q+\Delta) - \Phi(a,\zeta) + (\mu_D + a - \delta b)X)V_W - \kappa\zeta V_\zeta - \lambda b V_b \\ & + \left(\mathcal{B}_0 + \mathcal{B}_x X_t + \mathcal{B}_b b_t - \mathcal{B}_{\hat{b}}\hat{b}_t + \mathcal{B}_\zeta \zeta + \mathcal{B}_\Delta \Delta + \beta_D a\right) V_{\hat{b}} \\ & + (q+\Delta)V_x + \frac{1}{2}\left[\sigma_D^2 X^2 V_{WW} + \Sigma_{\hat{b}} V_{\hat{b}\hat{b}} + \sigma_b^2 V_{bb} + \sigma_\zeta V_{\zeta\zeta} + 2\sigma_D^2\beta_D X V_{W\hat{b}} + \beta_q\sigma_\zeta^2 V_{\hat{b}\zeta} + \beta_q\frac{Q_b}{Q_\zeta}\sigma_b^2 V_{\hat{b}b}\right]. \end{aligned}$$

Using the envelope condition, and evaluating at $\Delta = 0$, we get that

$$\begin{aligned} (r + \mathcal{B}_{\hat{b}})V_{\hat{b}} = & -R_{\hat{b}} q V_W + (rW - c - R(q, X, \hat{b})q - \Phi(a,\zeta) + (\mu_D + a - \delta b)X)V_{\hat{b}W} - \kappa\zeta V_{\hat{b}\zeta} - \lambda b V_{\hat{b}b} \\ & + \left(\mathcal{B}_0 + \mathcal{B}_x X_t + \mathcal{B}_b b_t - \mathcal{B}_{\hat{b}}\hat{b}_t + \mathcal{B}_\zeta \zeta + \mathcal{B}_\Delta \Delta + \beta_D a\right) V_{\hat{b}\hat{b}} \\ & + qV_{\hat{b}x} + \frac{1}{2}\left[\sigma_D^2 X^2 V_{\hat{b}WW} + \Sigma_{\hat{b}} V_{\hat{b}\hat{b}\hat{b}} + \sigma_b^2 V_{\hat{b}bb} + \sigma_\zeta V_{\hat{b}\zeta\zeta} + 2\sigma_D^2\beta_D X V_{\hat{b}W\hat{b}} + \beta_q\sigma_\zeta^2 V_{\hat{b}\hat{b}\zeta} + \beta_q\frac{Q_b}{Q_\zeta}\sigma_b^2 V_{\hat{b}\hat{b}b}\right]. \end{aligned}$$

Using the Feynman-Kac formula (Karatzas and Shreve, 2012), we get that

$$V_{\hat{b}}(W_t, X_t, \zeta_t, b_t, \hat{b}_t) = \mathbb{E}_t^L\left[\int_t^\infty e^{-(r+\mathcal{B}_{\hat{b}})(s-t)} R_{\hat{b}} q_s V_W(W_s, X_s, \zeta_s, b_s, \hat{b}_s) ds\right],$$

which means that

$$G_{\hat{b}}(X_t, \zeta_t, b_t, \hat{b}_t) = \mathbb{E}_t^M\left[\int_t^\infty e^{-(r+\mathcal{B}_{\hat{b}})(s-t)} R_{\hat{b}} q_s \frac{V_W(W_s, X_s, \zeta_s, b_s, \hat{b}_s, 0)}{V_W(W_t, X_t, \zeta_t, b_t, \hat{b}_t, 0)} ds\right].$$

Finally, using the first order condition for consumption $u_L'(c) = V_W$ we get

$$G_{\hat{b}}(X_t, \zeta_t, b_t, \hat{b}_t) = \mathbb{E}_t^L \left[\int_t^\infty e^{-(r+\mathcal{B}_{\hat{b}})(s-t)} R_{\hat{b}} q_s \frac{u_L'(c_s^L)}{u_L'(c_t^L)} ds \right]$$

□

B.1 System of Equations Equilibrium

The first step in the determination of the equilibrium is to determine the coefficients of the certainty equivalent. The system of equations determining the coefficients for the quadratic terms is decoupled from the system of equations determining the linear terms. After solving for the quadratic terms, we can determine the rest of the coefficients by solving a system of linear equations. Substituting the conjecture certainty equivalent H in Lemma 8 we find that

$$h_{xy} = \frac{r A_x}{r + Q_x} \tag{B.15a}$$

$$h_{y\hat{\zeta}} = \frac{A_\zeta + Q_\zeta h_{xy}}{r + \kappa} \tag{B.15b}$$

$$h_{y\hat{b}} = \frac{A_b + A_{\hat{b}} + h_{xy}\left(Q_b + Q_{\hat{b}}\right)}{r + \lambda} \tag{B.15c}$$

$$h_{yy} = -\frac{\gamma_M}{2}\left(\sigma_D^2 + \Sigma_{\hat{b}}^2 h_{y\hat{b}}^2 + \Sigma_{\hat{\zeta}}^2 h_{y\hat{\zeta}}^2 + 2\beta_D \sigma_D^2 \left(h_{y\hat{b}} - \frac{Q_b}{Q_\zeta} h_{y\hat{\zeta}}\right) + 2\Sigma_{\hat{b}\hat{\zeta}}^2 h_{y\hat{b}} h_{y\hat{\zeta}}\right). \tag{B.15d}$$

From the market clearing condition (21) we get the pricing coefficients

$$P_0 = h_y + 2h_{yy} \tag{B.15ea}$$

$$P_x = h_{yx} - 2h_{yy} \tag{B.15eb}$$

$$P_{\hat{\zeta}} = h_{y\hat{\zeta}} \tag{B.15ec}$$

$$P_{\hat{b}} = h_{y\hat{b}}. \tag{B.15ed}$$

The coefficients of the residual supply are

$$R_0 = P_0 - P_{\hat{\zeta}} \frac{Q_0}{Q_\zeta} \tag{B.15fa}$$

$$R_x = P_x + P_{\hat{\zeta}} \frac{Q_x}{Q_\zeta} \tag{B.15fb}$$

$$R_{\hat{b}} = P_{\hat{b}} - \frac{Q_b + Q_{\hat{b}}}{Q_\zeta} P_{\hat{\zeta}} \tag{B.15fc}$$

$$R_q = \frac{P_{\hat{\zeta}}}{Q_\zeta}. \tag{B.15fd}$$

The next step, is to derive a system of equations for the coefficient of the certainty equivalent G. To simplify the notation, let's introduce a 5×1 vector containing the state variables $z \equiv (X, \zeta, b, \hat{b}, \Delta)$, and write and

$$G(X, \zeta, b, \hat{b}, \Delta) \equiv g_0 + \mathbf{g}_z^\intercal z + z^\intercal \mathbf{G}_{zz} z,$$

where g_0 is a scalar, $\mathbf{g}_z \equiv (g_x, g_\zeta, g_b, g_{\hat{b}}, g_\Delta)^\intercal$ is a 5×1 vector, and $\mathbf{G}_{zz}$ is the following 5×5 symmetric matrix

$$\mathbf{G}_{zz} = \begin{pmatrix} g_{xx} & \frac{1}{2}g_{x\zeta} & \frac{1}{2}g_{xb} & \frac{1}{2}g_{x\hat{b}} & \frac{1}{2}g_{x\Delta} \\ \frac{1}{2}g_{x\zeta} & g_{\zeta\zeta} & \frac{1}{2}g_{\zeta b} & \frac{1}{2}g_{\zeta\hat{b}} & \frac{1}{2}g_{\zeta\Delta} \\ \frac{1}{2}g_{xb} & \frac{1}{2}g_{\zeta b} & g_{bb} & \frac{1}{2}g_{b\hat{b}} & \frac{1}{2}g_{b\Delta} \\ \frac{1}{2}g_{x\hat{b}} & \frac{1}{2}g_{\zeta\hat{b}} & \frac{1}{2}g_{b\hat{b}} & g_{\hat{b}\hat{b}} & \frac{1}{2}g_{\hat{b}\Delta} \\ \frac{1}{2}g_{x\Delta} & \frac{1}{2}g_{\zeta\Delta} & \frac{1}{2}g_{b\Delta} & \frac{1}{2}g_{\hat{b}\Delta} & g_{\Delta\Delta} \end{pmatrix}$$

Let $\mathbf{1}_i$ vector 5×1 vector with a one in the i-th row and zeros in the remaining entries, and $\mathbf{1}_{i,j} \equiv \mathbf{1}_i \mathbf{1}_j^\intercal$ be a 5×5 matrix with a one in the ij-th entry and zeros in the remaining entries. Let $A_\zeta \equiv (A_x, A_\zeta, A_b, A_{\hat{b}}, A_\Delta)^\intercal$ and $Q_\zeta \equiv (Q_x, Q_\zeta, Q_b, Q_{\hat{b}}, 0)^\intercal$ be 5×1 vectors with the coefficients of the effort and trading strategies. From the first order condition for effort we get that

$$A_0 = \frac{\beta_D}{2\phi} \mathbf{1}_4^\intercal \mathbf{g}_z \tag{B.7a}$$

$$A_z = \frac{1}{2\phi} \left[\mathbf{1}_1 + \psi \mathbf{1}_2 + 2\beta_D \mathbf{1}_4^\intercal \mathbf{G}_{zz} \mathbf{1}_4 \right] \tag{B.7b}$$

Let $R_z \equiv (R_x, R_\zeta, R_b, R_{\hat{b}}, 0)^\intercal$ and $\mathcal{B}_z \equiv (\mathcal{B}_x, \mathcal{B}_\zeta, \mathcal{B}_b, \mathcal{B}_{\hat{b}}, \mathcal{B}_\Delta)^\intercal$, where the coefficients in $\mathcal{B}_z$ are given in (B.4)-(B.6). Substituting in the HJB equation for the certainty equivalent in Proposition 9 and

matching coefficients we get

$$rg_0 = -(R_0 + R_q Q_0)Q_0 - \frac{r\gamma_L}{2}\mathbf{g}_z^\intercal \left[\Sigma_{\hat{b}}\mathbf{1}_{4,4} + \sigma_b^2\mathbf{1}_{3,3} + \sigma_\zeta \mathbf{1}_{2,2} + \beta_q\sigma_\zeta^2\mathbf{1}_{2,4} + \beta_q\frac{Q_b}{Q_\zeta}\sigma_b^2\mathbf{1}_{3,4}\right]\mathbf{g}_z \tag{B.8a}$$

$$- \phi A_0^2 + (\mu_0 + \beta_D A_0)\mathbf{1}_4^\intercal\mathbf{g}_z + Q_0\mathbf{1}_1^\intercal\mathbf{g}_z + \left[\Sigma_{\hat{b}}\mathbf{1}_{4,4} + \sigma_b^2\mathbf{1}_{3,3} + \sigma_\zeta\mathbf{1}_{2,2} + \beta_q\sigma_\zeta^2\mathbf{1}_{2,4} + \beta_q\frac{Q_b}{Q_\zeta}\sigma_b^2\mathbf{1}_{3,4}\right]\mathbf{G}_{zz}$$

$$r\mathbf{g}_z^\intercal = (\mu_D + A_0)\mathbf{1}_1^\intercal - Q_0 R_z^\intercal - (R_0 + 2R_q Q_0)(Q_z + \mathbf{1}_5)^\intercal - 2\phi A_0 A_z^\intercal + \psi A_0\mathbf{1}_2^\intercal \tag{B.8b}$$

$$- r\gamma_L\sigma_D^2\beta_D\mathbf{g}_z^\intercal\mathbf{1}_{3,1} - 2r\gamma_L\mathbf{g}_z^\intercal\left[\Sigma_{\hat{b}}\mathbf{1}_{4,4} + \sigma_b^2\mathbf{1}_{3,3} + \sigma_\zeta\mathbf{1}_{2,2} + \beta_q\sigma_\zeta^2\mathbf{1}_{2,4} + \beta_q\frac{Q_b}{Q_\zeta}\sigma_b^2\mathbf{1}_{3,4}\right]\mathbf{G}_{zz}$$

$$+ 2Q_0\mathbf{1}_1^\intercal\mathbf{G}_{zz} + \mathbf{g}_z^\intercal(Q_z^\intercal\mathbf{1}_1 + \mathbf{1}_{1,5}) - (\kappa\mathbf{1}_{2,2} + \lambda\mathbf{1}_{3,3})\,\mathbf{g}_z^\intercal + 2(\mathcal{B}_0 + \beta_D A_0)\mathbf{1}_4^\intercal\mathbf{G}_{zz}$$

$$+ \mathbf{g}_z^\intercal\mathbf{1}_4(\mu_z + \beta_D A_z)^\intercal$$

$$r\mathbf{G}_{zz} = A_z\mathbf{1}_1^\intercal - \delta\mathbf{1}_{3,1} - \big(R_z + R_q(Q_z + \mathbf{1}_5)\big)(Q_z + \mathbf{1}_5)^\intercal - \phi A_z A_z^\intercal + \psi A_z\mathbf{1}_2^\intercal \tag{B.8c}$$

$$- \frac{r\gamma_L}{2}\left[\sigma_D^2\mathbf{1}_{1,1} + 4\sigma_D^2\beta_D\mathbf{G}_{zz}\mathbf{1}_{3,1}\right] - 2\left(\kappa\mathbf{1}_{2,2} + \lambda\mathbf{1}_{3,3}\right)\mathbf{G}_{zz} + 2(\mathcal{B}_z + \beta_D A_z)\mathbf{1}_4^\intercal\mathbf{G}_{zz}$$

$$- 2r\gamma_L\mathbf{G}_{zz}\left[\Sigma_{\hat{b}}\mathbf{1}_{4,4} + \sigma_b^2\mathbf{1}_{3,3} + \sigma_\zeta^2\mathbf{1}_{2,2} + \beta_q\sigma_\zeta^2\mathbf{1}_{2,4} + \beta_q\frac{Q_b}{Q_\zeta}\sigma_b^2\mathbf{1}_{3,4}\right]\mathbf{G}_{zz} + 2(Q_z\mathbf{1}_1^\intercal + \mathbf{1}_{5,1})\mathbf{G}_{zz}$$

Finally, we consider the first order condition determining the coefficients in Q_z, which is given by

$$\left(\mathbf{1}_5^\intercal + \frac{\beta_q}{Q_\zeta}\mathbf{1}_4^\intercal\right)\mathbf{g}_z = 0 \tag{B.9a}$$

$$\mathbf{J}\mathbf{G}_{zz}\left(\mathbf{1}_5 + \frac{\beta_q}{Q_\zeta}\mathbf{1}_4\right) = 0 \tag{B.9b}$$

where

$$\mathbf{J} \equiv \begin{pmatrix} \mathbf{I}_{4\times4} & \mathbf{O}_{4\times1} \\ \mathbf{O}_{1\times4} & 0 \end{pmatrix}$$

and $\mathbf{I}_{4\times4}$ is a 4×4 identity matrix and $\mathbf{O}_{n\times m}$ is a $n\times m$ matrix of zeros. Thus, to find an equilibrium, we need to solve the system given by equations (B.3) and (B.15a)-(B.9b).

B.2 Impulse Response Functions

In order to compute the impulse response function we use the following results that can be found in Evans (2012).

Lemma B.1. *The solution to the linear SDE*

$$d\mathbf{X}_t = (\mathbf{c} + \mathbf{D}\mathbf{X}_t)dt + \mathbf{E}d\mathbf{W}_t.$$

is

$$\mathbf{X}_t = e^{\mathbf{D}t}\mathbf{X}_0 + \int_0^t e^{\mathbf{D}(t-s)}(\mathbf{c}ds + \mathbf{E}d\mathbf{W}_s),$$

where $e^{\mathbf{D}t}$ *is the matrix exponential.*

Next we derive the impulse response function. We start deriving the impulse response functions under $\mathcal{F}_t^M$. The blochholder block size is determined by the solution to the following linear system of stochastic differential equation

$$\begin{pmatrix} dX_t \\ d\hat{\zeta}_t \\ d\hat{b}_t \end{pmatrix} = \left[\begin{pmatrix} Q_0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} -Q_x & Q_\zeta & Q_b + Q_{\hat{b}} \\ 0 & -\kappa & 0 \\ 0 & 0 & -\lambda \end{pmatrix} \begin{pmatrix} X_t \\ \hat{\zeta}_t \\ \hat{b}_t \end{pmatrix} \right] dt + \begin{pmatrix} 0 & 0 & 0 \\ \sigma_\zeta \left(1 - \frac{Q_b}{Q_\zeta}\beta_q\right) & \sigma_b \frac{Q_b}{Q_\zeta}\left(1 - \beta_q \frac{Q_b}{Q_\zeta}\right) & -\sigma_D \beta_D \frac{Q_b}{Q_\zeta} \\ \sigma_\zeta \beta_q & \sigma_b \beta_q \frac{Q_b}{Q_\zeta} & \sigma_D \beta_D \end{pmatrix} \begin{pmatrix} d\tilde{B}_t^\zeta \\ d\tilde{B}_t^b \\ d\tilde{B}_t^D \end{pmatrix}.$$

The solution to this equation is (see, e.g. Evans (2012))

$$\begin{pmatrix} X_t \\ \hat{\zeta}_t \\ \hat{b}_t \end{pmatrix} = \Pi(t) \begin{pmatrix} X_0 \\ \hat{\zeta}_0 \\ \hat{b}_0 \end{pmatrix} + \int_0^t \Pi(t-s) \left[\begin{pmatrix} Q_0 \\ 0 \\ 0 \end{pmatrix} ds + \begin{pmatrix} 0 & 0 & 0 \\ \sigma_\zeta \left(1 - \frac{Q_b}{Q_\zeta}\beta_q\right) & \sigma_b \frac{Q_b}{Q_\zeta}\left(1 - \beta_q \frac{Q_b}{Q_\zeta}\right) & -\sigma_D \beta_D \frac{Q_b}{Q_\zeta} \\ \sigma_\zeta \beta_q & \sigma_b \beta_q \frac{Q_b}{Q_\zeta} & \beta_D \sigma_D \end{pmatrix} \begin{pmatrix} d\tilde{B}_s^\zeta \\ d\tilde{B}_s^b \\ d\tilde{B}_s^D \end{pmatrix} \right],$$

where

$$\Pi(t) = \begin{pmatrix} e^{-Q_x t} & \frac{\left(e^{-\kappa t} - e^{-Q_x t}\right)Q_\zeta}{Q_x - \kappa} & \frac{\left(e^{-\lambda t} - e^{-Q_x t}\right)(Q_b + Q_{\hat{b}})}{Q_x - \lambda} \\ 0 & e^{-\kappa t} & 0 \\ 0 & 0 & e^{-\lambda t} \end{pmatrix}.$$